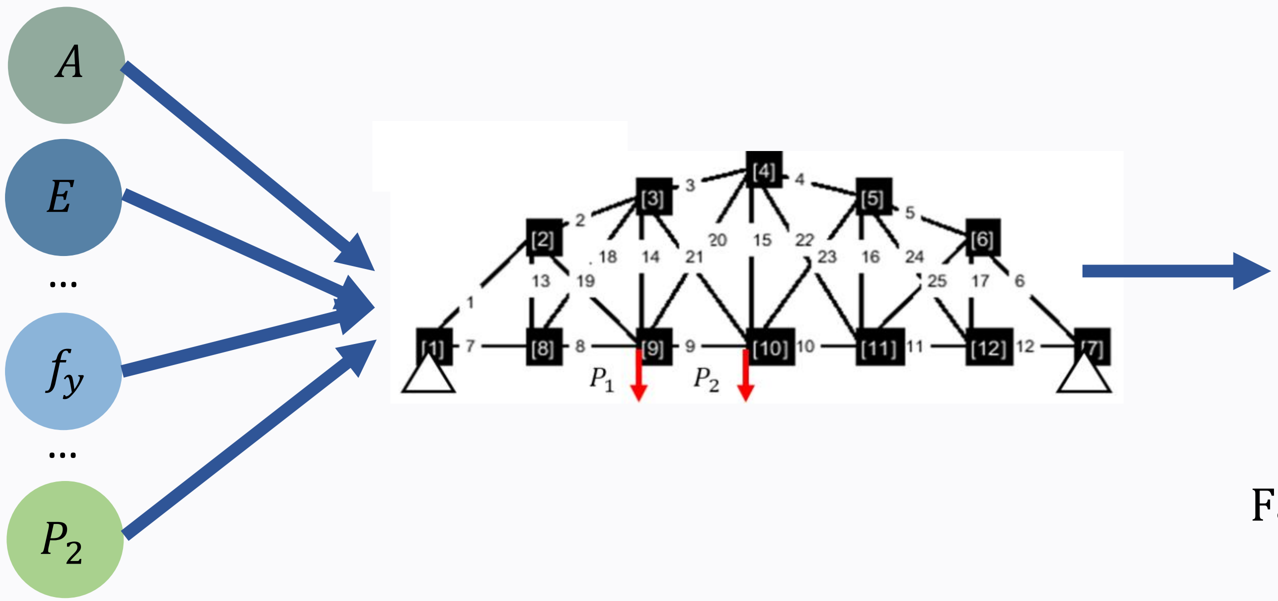
Global Sensitivity Analysis

Motivation

- Consider a numerical model of a structure

Inputs **Model** **Output**



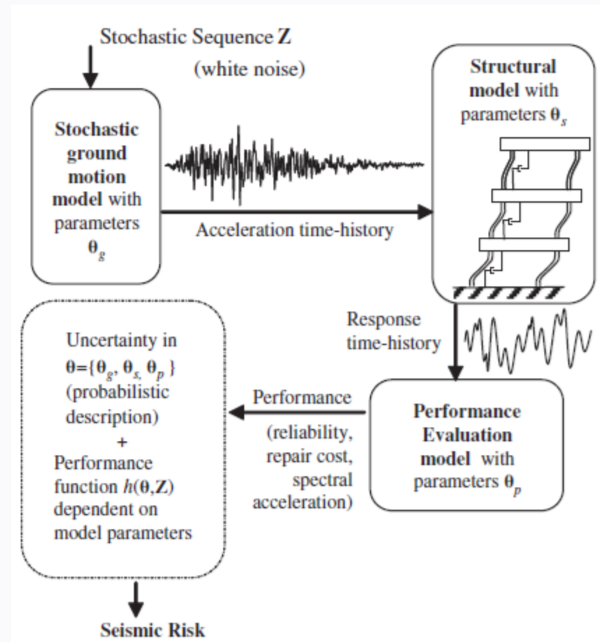
midspan displacement

Failure index (0,1)

- Will all the input parameters contribute to the response?
- Which input factors are more influential than others?

Applications of GSA

- **To gain insights**
  - How different parameters and their interactions affect a system



The parameters related to **the local site diminution effect** (…) can be **in general neglected;** (…) **temporal envelope function parameters** have **a considerable contribution** towards the total risk only for lower moment magnitudes, especially (…)
(Vetter & Taflanidis, 2011)

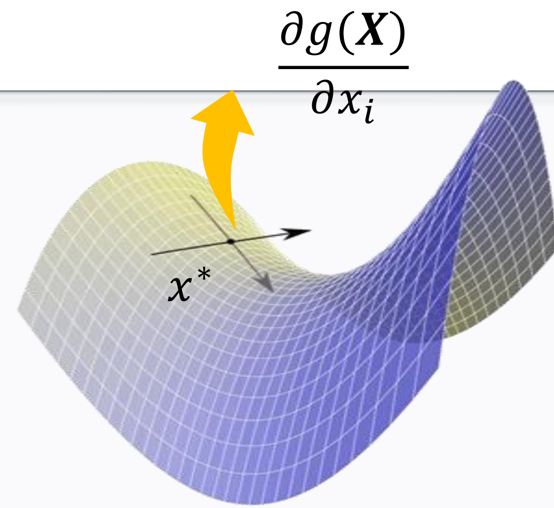
Applications of GSA

- **To gain insights**
  - How different parameters and their interactions affect a system
- **Dimensionality reduction**
  - By identifying uninfluential (redundant) factors
- **Informed decision making**
  - To find parameters for which new data acquisition reduces target uncertainty the most
  - To identify most effective decision options
- **Model diagnostics**
  - After developing a model, one may compare GSA results with expert knowledge

Local Sensitivity Analysis

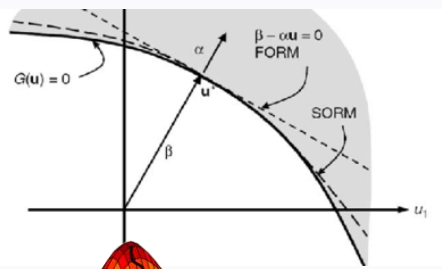
- Rate of change (slope)

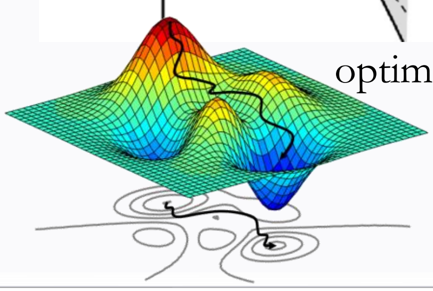
$$S_i^D(\boldsymbol{X}) = \frac{\partial g(\boldsymbol{X})}{\partial X_i}$$



- Studies the impact of small perturbations on the model outputs
- Evaluated at a reference point $\boldsymbol{X}$
- One-factor-at-a-time evaluation
- Used in reliability analysis / optimization

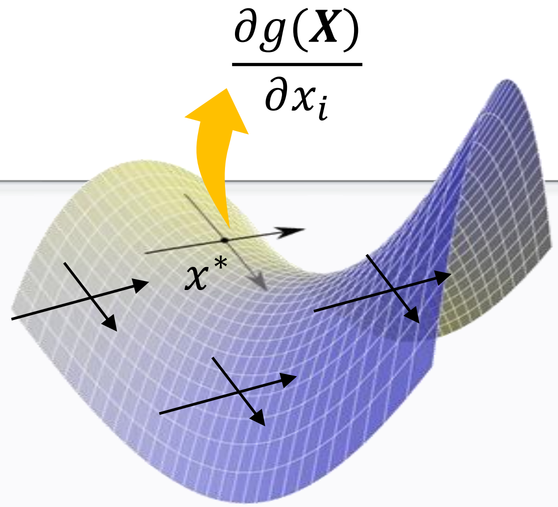
e.g. structural reliability analysis





optimization

Local Sensitivity Analysis



- How to explore entire variability space?

Is the average of gradients a good measure?

- Consider an example

$$Y = g(X_1, X_2) = 2X_1 + X_2$$

$$X_1 \sim N(0, 0.5^2), X_2 \sim N(0, 5^2)$$

Among X_1 and X_2, which variable is more "important"?

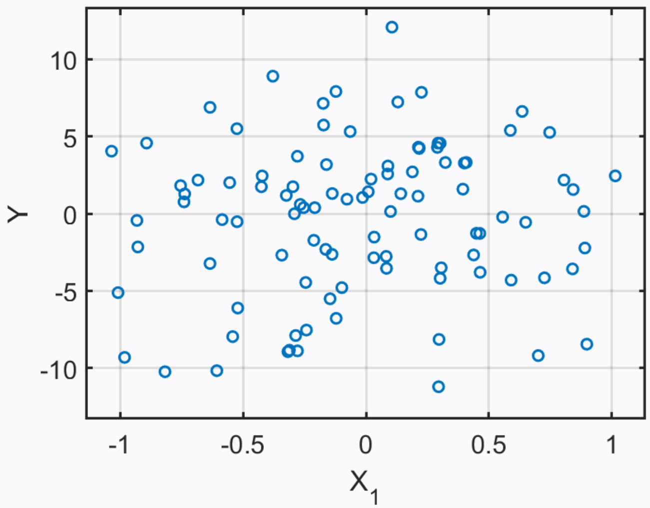
Local Sensitivity Analysis

$$Y = g(X_1, X_2) = 2X_1 + X_2$$

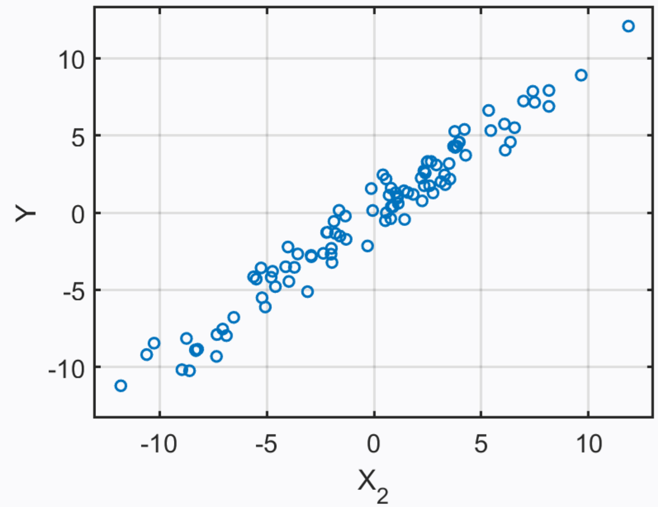
$$X_1 \sim N(0, 0.5^2), \quad X_2 \sim N(0, 5^2)$$

- If we decide the importance by 'partial derivative' measure, X_1 is important
- But if we inspect the scatter plots,

X_2 seem to dominate the response



Y vs. X_1



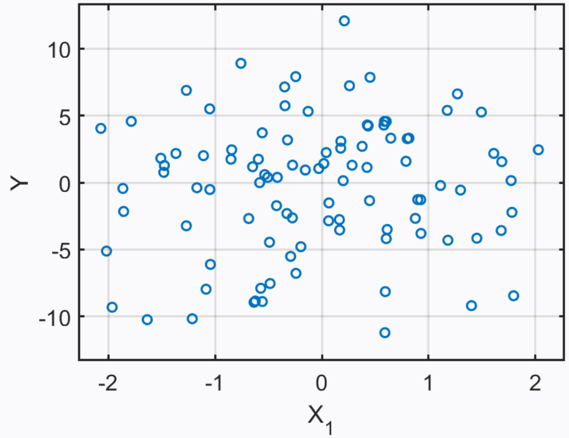
Y vs. X_2

Local Sensitivity Analysis

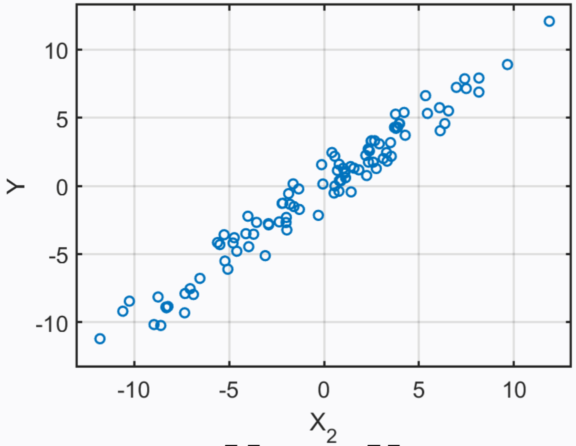
- Sigma-normalized derivative

$$S_i^{SD}(\boldsymbol{X}) = \frac{\sigma_{X_i}}{\sigma_Y}\frac{\partial g(\boldsymbol{X})}{\partial X_i}$$

X_2 is five times more important than X_1



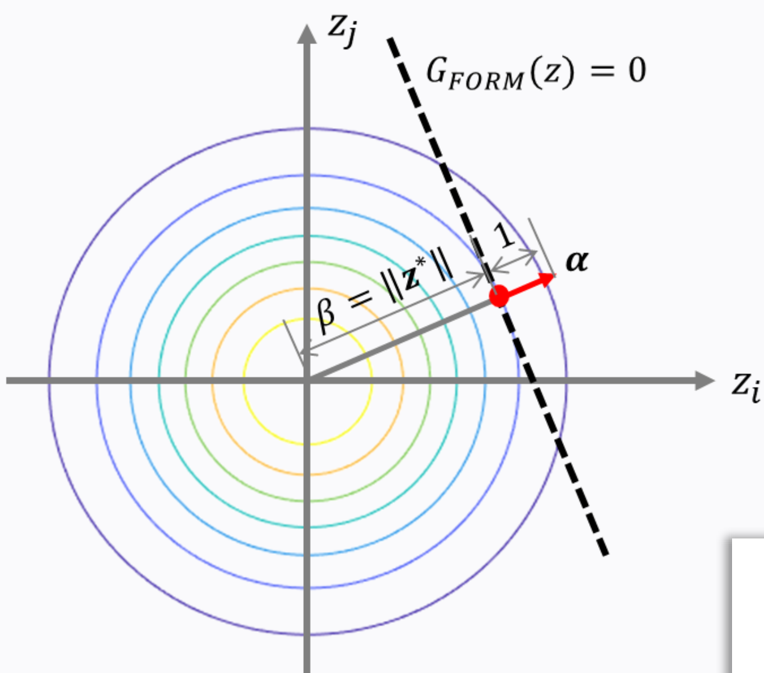
Y vs. X_1



Y vs. X_2

Local Sensitivity Analysis

- 'Partial derivative' in the standard random variable domain
  - When the random variables are **independent**, each variable can be transformed to the standard normal variable, $Z_i = T(X_i)$.
  - Example: FORM analysis



Importance vector:

normalized gradient

$$\boldsymbol{\alpha} = -\frac{\nabla G(\boldsymbol{z}^*)}{\|\nabla G(\boldsymbol{z}^*)\|}$$

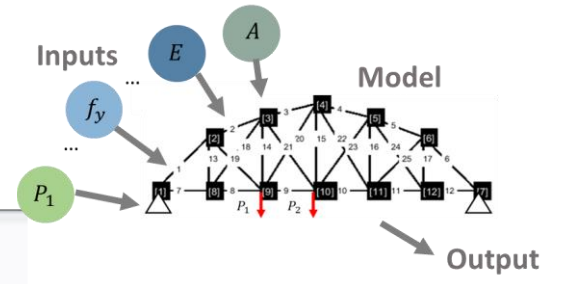
Note

$$\boldsymbol{\alpha} = -\frac{\boldsymbol{z}^*}{\beta}$$

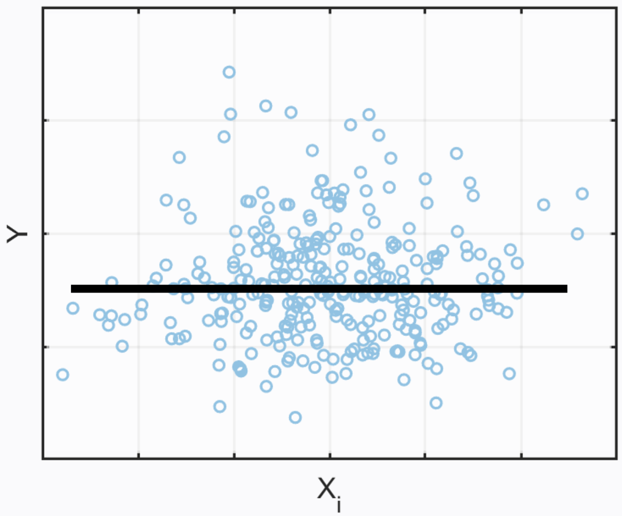
$$Z_1 = F_Z^{-1}\left(F_{X_1}(X_1)\right)$$
$$Z_2 = F_Z^{-1}\left(F_{X_2|X_1}(X_2|X_1)\right)$$
$$\vdots$$
$$Z_n = F_Z^{-1}\left(F_{X_n|X_{n-1},\ldots,X_1}(X_n|X_{n-1},\ldots,X_1)\right)$$

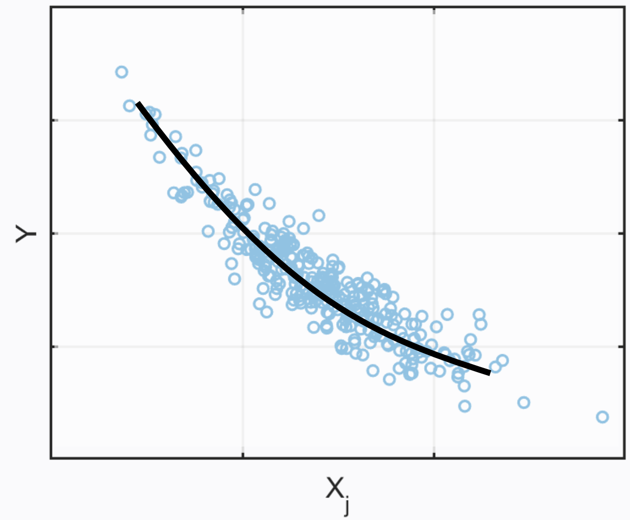
For dependent variables

Variance-based Sensitivity



- Intuition behind the Sobol indices





$\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]$ is almost constant throughout different x_i values

$\Rightarrow$ $\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]$ is almost zero

$\Rightarrow$ Low sensitivity

$\mathbb{E}_{\mathbf{x}_{\bar{j}}}[\mathrm{Y}|x_j]$ depends on x_j

$\Rightarrow$ $\mathbb{V}\mathrm{ar}_{x_j}\left[\mathbb{E}_{\mathbf{x}_{\bar{j}}}[\mathrm{Y}|x_j]\right]$ is larger

$\Rightarrow$ High sensitivity

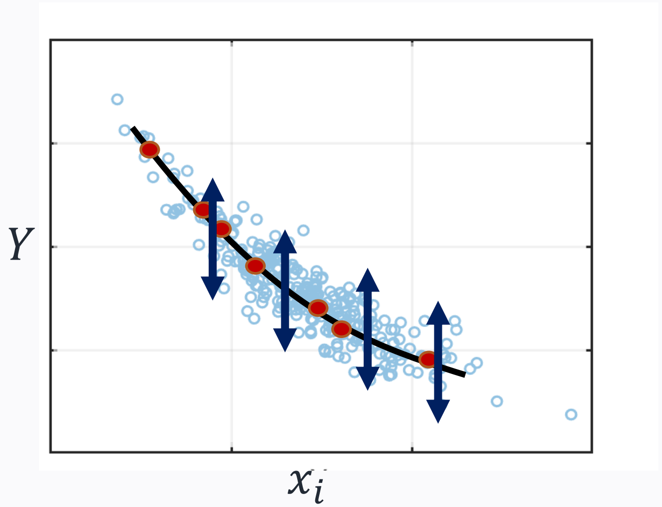
Variance Decomposition

- $\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[Y|x_i]\right]$ is a measure of sensitivity
- The Law of Total Variance

$$\mathbb{V}\mathrm{ar}[Y] = \mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[Y|x_i]\right] + \mathbb{E}_{x_i}\left[\mathbb{V}\mathrm{ar}_{\mathbf{x}_{\bar{i}}}[Y|x_i]\right]$$

Explained by x_i **Not explained by x_i**

i.e. the expected reduction in variance that would be obtained if x_i could be fixed



Variance Decomposition

- $\mathbb{V}\text{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i]\right]$ is a measure of sensitivity
- The Law of Total Variance

$$\mathbb{V}\text{ar}[\text{Y}] = \mathbb{V}\text{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i]\right] + \mathbb{E}_{x_i}\left[\mathbb{V}\text{ar}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i]\right]$$

- Derivation

$$\begin{aligned}
\mathbb{V}\text{ar}[\text{Y}] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[Y^2|x_i]\right] - \mathbb{E}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[Y|x_i]\right]^2 \\
&= \mathbb{E}_{x_i}\left[\mathbb{V}\text{ar}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i] + \mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[Y|x_i]^2\right] - \mathbb{E}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i]\right]^2 \\
&= \mathbb{E}_{x_i}\left[\mathbb{V}\text{ar}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i]\right] + \mathbb{E}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[Y|x_i]^2\right] - \mathbb{E}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[Y|x_i]\right]^2 \\
&= \mathbb{E}_{x_i}\left[\mathbb{V}\text{ar}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i]\right] + \mathbb{V}\text{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{\imath}}}[\text{Y}|x_i]\right]
\end{aligned}$$

Variance Decomposition

- $\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]$ is a measure of sensitivity
- The Law of Total Variance

$$\mathbb{V}\mathrm{ar}[\mathrm{Y}] = \mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right] + \mathbb{E}_{x_i}\left[\mathbb{V}\mathrm{ar}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]$$

Always greater than 0

$$1 = \frac{\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]}{\mathbb{V}\mathrm{ar}[\mathrm{Y}]} + \frac{\mathbb{E}_{x_i}\left[\mathbb{V}\mathrm{ar}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]}{\mathbb{V}\mathrm{ar}[\mathrm{Y}]}$$

Sensitivity index

In range of [0,1]

Variance Decomposition

- The Law of Total Variance

$$1 = \frac{\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]}{\mathbb{V}\mathrm{ar}[\mathrm{Y}]} + \frac{\mathbb{E}_{x_i}\left[\mathbb{V}\mathrm{ar}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]}{\mathbb{V}\mathrm{ar}[\mathrm{Y}]}$$

- Sobol Sensitivity Index

$$S_i = \frac{\mathbb{V}\mathrm{ar}\left[\mathbb{E}[\mathrm{Y}|x_i]\right]}{\mathbb{V}\mathrm{ar}[\mathrm{Y}]}$$

$$S_i = 1 - \frac{\mathbb{E}\left[\mathbb{V}\mathrm{ar}[\mathrm{Y}|x_i]\right]}{\mathbb{V}\mathrm{ar}[\mathrm{Y}]}$$

- Main-effect index, First-order index

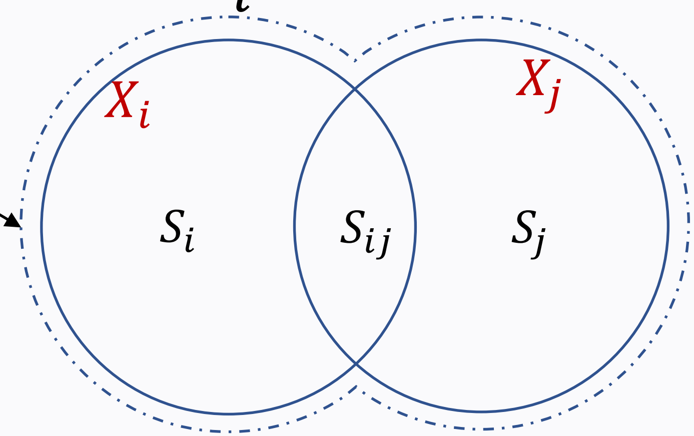
Second-order Sensitivity Measures

$$S_{ij} = \frac{\mathbb{V}\mathrm{ar}_{x_i x_j}\left[\mathbb{E}_{\mathbf{x}_{\overline{ij}}}[Y|X_i, X_j]\right]}{\mathbb{V}\mathrm{ar}[Y]} - S_i - S_j$$

contribution of X_j

joint contribution of X_i and X_j

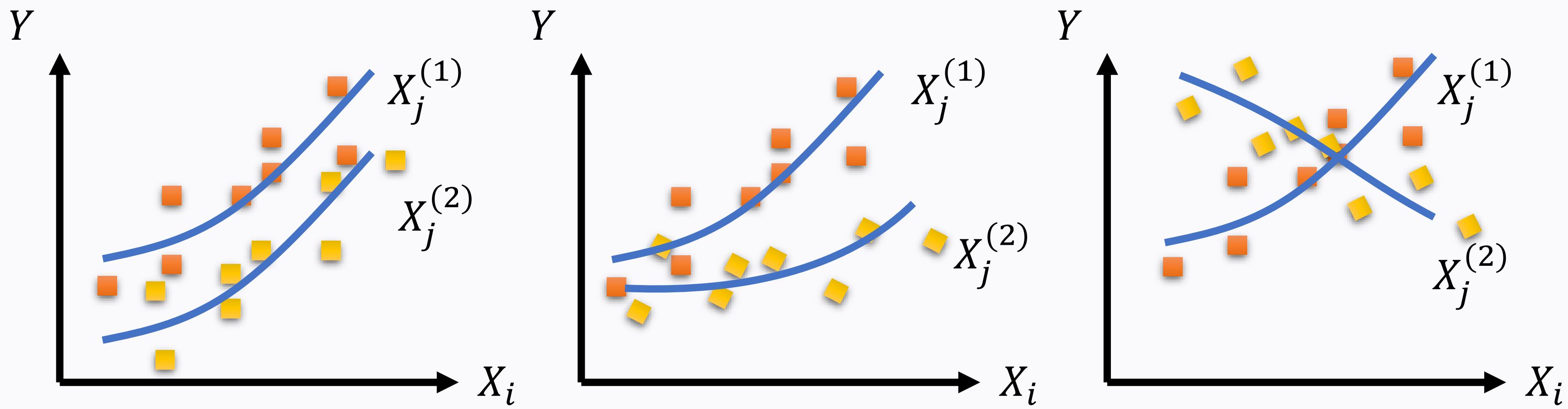
contribution of X_i



S_{ij} captures the pure interaction effect

Interaction Effect

- Interaction effect: X_i vs. Y is affected by X_j



No interaction
$S_{ij} = 0$

Interaction between X_i and X_j
$S_{ij} > 0$

- Nonadditive terms create the interaction

$$g_A(X_1, X_2) = 3X_1^3 + \log(X_2)$$

No interaction $S_{12} = 0$

$$g_B(X_1, X_2) = 3X_1^3 + \log(X_2) + X_1 X_2$$

Interaction between X_1 and X_2

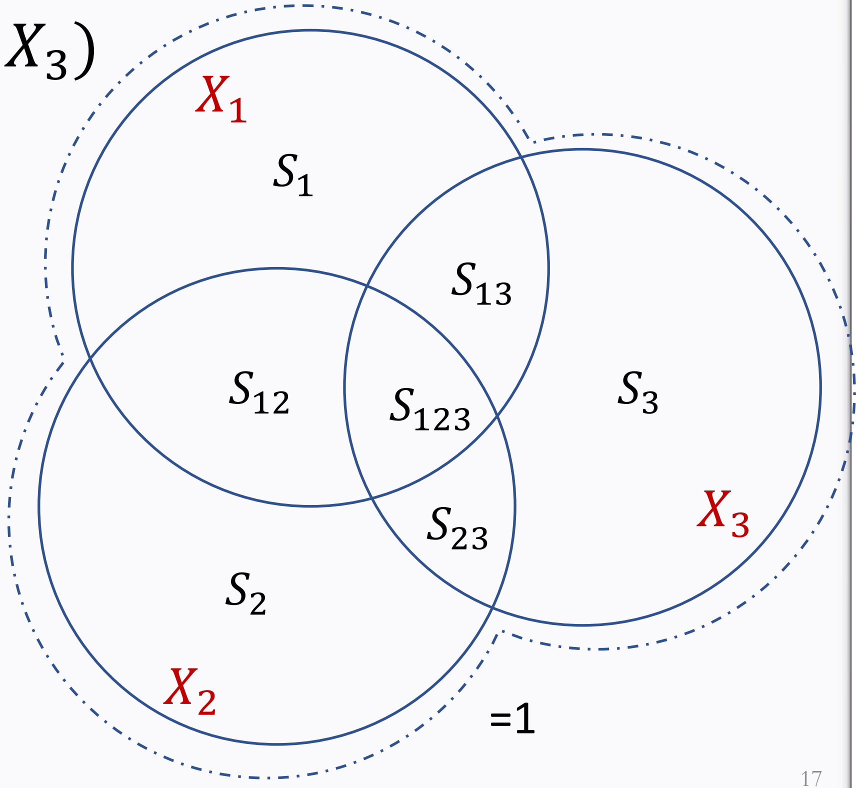
$S_{12} > 0$

Higher-order Sensitivity Indices

When random variables are independent below holds

$$1 = \sum_i S_i + \sum_{i<j} S_{ij} + \ldots + S_{1,2,\ldots,d}$$

Consider an example $Y = g(X_1, X_2, X_3)$



Total-effect Index

$$S_i^T = 1 - \frac{\mathbb{V}\mathrm{ar}_{\boldsymbol{X}_{\bar{\imath}}}\left[\mathbb{E}_{x_i}[Y|\boldsymbol{X}_{\bar{\imath}}]\right]}{\mathbb{V}\mathrm{ar}[Y]}$$

Conditioning on all variables but X_i

S_i^T accounts for all the interaction effects associated with a variable X_i

Total-effect Index

- For example, consider a function

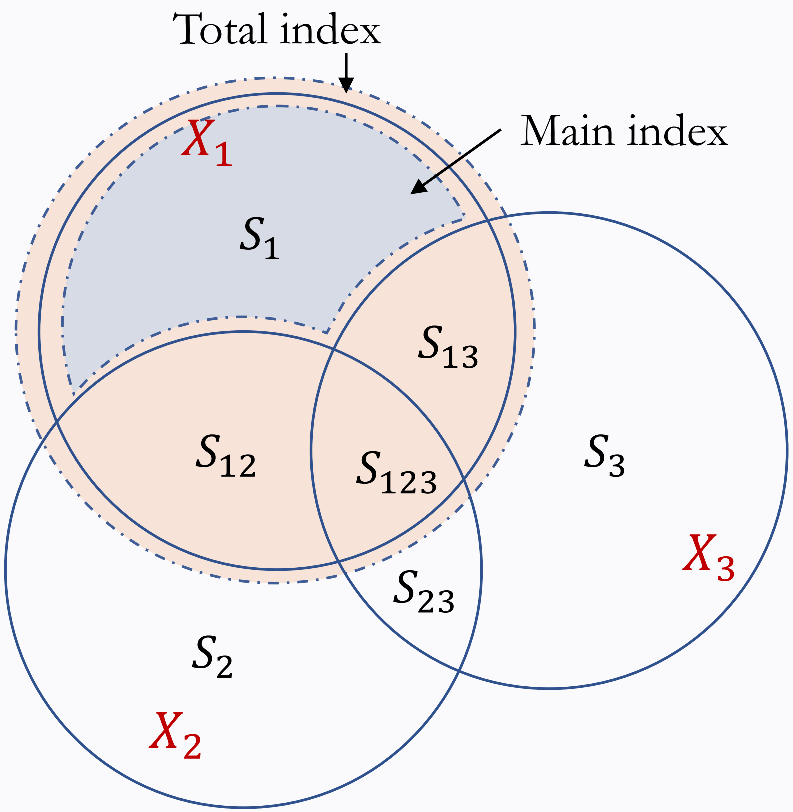
$$Y = g(X_1, X_2, X_3)$$

Total-effect index for X_1 is

$$S_1^T = 1 - S_{23} - S_2 - S_3$$

When the variables are independent

$$S_1^T = S_1 + S_{12} + S_{13} + S_{123}$$



Analysis of Variance (ANOVA) Decomposition

Consider uncorrelated $\boldsymbol{X}$ distributed within a unit hyper-cube

$$Y = g(\boldsymbol{X})$$

The function can be expanded as

$$Y = g_0 + \sum_{i} g_i(X_i) + \sum_{i<j} g_{ij}(X_i, X_j) + g_{12,\dots,d}(X_{1,} X_{2,} \dots , X_d)$$

This formula is called ANOVA representation if

$$\int_0^1 g_{\boldsymbol{u}}(\boldsymbol{X}_{\boldsymbol{u}}) dx_k = 0, \qquad k \in \boldsymbol{u}$$

for any $\boldsymbol{u} \subseteq \{1,2, \dots , d\}$. For example,

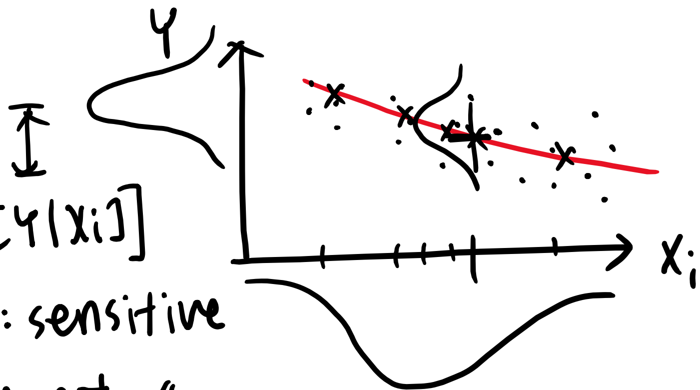
$$\int_0^1 g_{ij}(X_i, X_j) dX_i = 0 \qquad \text{and} \qquad \int_0^1 g_{ij}(X_i, X_j) dX_j = 0$$

Review

1) main sobol index

$$S_i = \frac{Var[E[Y|X_i]]}{Var[Y]}$$

$$\boxed{S_i \leq S_i^T}$$



$E[Y|\underline{X_i}]$

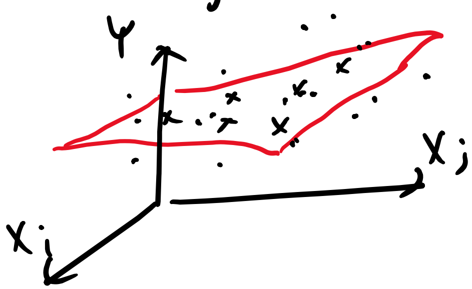
$Var[E[Y|X_i]]$

- large: sensitive
- small: not 〃

2) higher order sobol index

$$S_{ij} = \frac{Var[E[Y|X_i, X_j]]}{Var[Y]} - S_i - S_j$$

↖ includes 'effect' of X_i, X_j, interaction b/w X_i, X_j



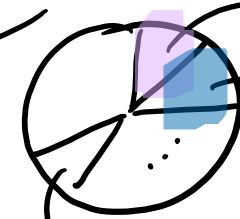
3) total sobol index

$$S_i^T = 1 - \frac{Var[E[Y|\underline{X_{\sim i}}]]}{Var[Y]}$$

e.g. when $d=3$ $\quad S_2^T = S_2 + S_{12} + S_{23} + S_{123}$

ANOVA - derivation

$y = f(x_1, x_2, \cdots, x_d)$

$Var[y] =$ 

assump. $X \sim uniform(0,1)$, ind.

$$Y = g(x_1, x_2, \cdots, x_d)$$

$$= g_0 + \sum_i g_i(x_i) + \sum_{i<j} g_{ij}(x_i, x_j) + \cdots g_{12\cdots d}(x_1, \cdots, x_d)$$

given orthogonality ↳ is unique — "ANOVA".

e.g.

$$\int_0^1 g_{235}(x_2, x_3, x_5)\, dx_2 = 0$$

$$\int_0^1 g_{235}(x_2, x_3, x_5)\, dx_3 = 0$$

$$dx_5 = 0$$

$$S_i = \frac{V_i}{V}$$

sensitivity

$$Var[Y] = V = \sum_i V_i + \sum_{i<j} V_{ij} + \cdots + V_{12\cdots d}.$$

V_i: Variance fraction coming from X_i

$\sum_{i<j} V_{ij} + \cdots + V_{12\cdots d}$: interaction.

→ 50% x_1

→ 40% x_2

→ 10% x_1 & x_2.

Because of orthogonality

$$E[Y] = g_0$$

$$E[Y|X_i] = g_0 + g_i(X_i)$$

$$E[Y|X_i,X_j] = g_0 + g_i(X_i) + g_j(X_j) + g_{ij}(X_i,X_j)$$

↳

$$g_i(X_i) = E[Y|X_i] - g_0$$

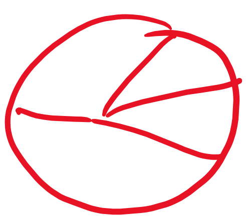
$$g_{ij}(X_i,X_j) = E[Y|X_i,X_j] - E[Y|X_i] - E[Y|X_j] + g_0$$

↳ (when x independent)

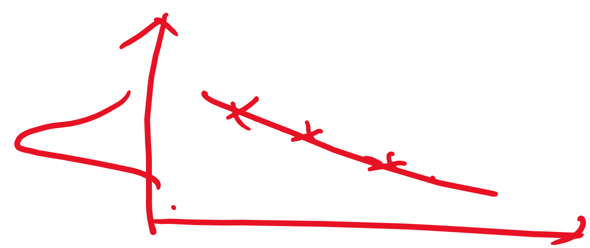
$$\frac{Var[g_i(X_i)]}{Var[Y]} = \frac{V_i}{V} = \frac{Var[E[Y|X_i]]}{Var[Y]} = S_i$$

ANOVA.

Law of total var



decomposition of total var;



Analysis of Variance (ANOVA) Decomposition

$$Y = g_0 + \sum_i g_i(X_i) + \sum_{i<j} g_{ij}(X_i, X_j) + g_{12,\ldots,d}(X_1, X_2, \ldots, X_d)$$

Taking $Var[\cdot]$ on both sides

$$\mathbb{V}\mathrm{ar}[Y] = V = \sum_i V_i + \sum_{i<j} V_{ij} + \ldots + V_{12..d}$$

The proportion of variance attributed to X_i

$$1 = \sum_i \frac{V_i}{V} + \sum_{i<j} \frac{V_{ij}}{V} + \ldots + \frac{V_{12..d}}{V}$$

Equivalent to Sobol index why?

Analysis of Variance (ANOVA) Decomposition

$$Y = g_0 + \sum_i g_i(X_i) + \sum_{i<j} g_{ij}(X_i, X_j) + g_{12,\dots,d}(X_1, X_2, \dots, X_d)$$

=0

$\mathbb{E}[\cdot]$ on both sides , i.e. integrate over $\boldsymbol{X}$

$$\mathbb{E}[Y] = g_0$$

$\mathbb{E}[\cdot\,|\boldsymbol{X_u}]$ on both sides, i.e. integrate over all but $\boldsymbol{u} \subseteq \{1,2,\dots,d\}$

$$\mathbb{E}_{\mathbf{x}_{\bar{i}}}[Y|X_i] = g_0 + g_i(X_i)$$

$$\mathbb{E}_{\mathbf{x}_{\bar{i}}}\left[Y\middle|X_i, X_j\right] = g_0 + g_i(X_i) + g_j(X_j) + g_{ij}(X_i, X_j)$$

….

Analysis of Variance (ANOVA) Decomposition

$$\mathbb{E}_{\mathbf{x}_{\bar{i}}}[Y|X_i] = g_0 + g_i(X_i)$$

$$\mathbb{E}_{\mathbf{x}_{\bar{i}}}\left[Y\middle|X_i, X_j\right] = g_0 + g_i(X_i) + g_j\left(X_j\right) + g_{ij}\left(X_i, X_j\right)$$

....

$$\mathbb{V}\mathrm{ar}_{x_i}[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[Y|X_i]] = V_i$$

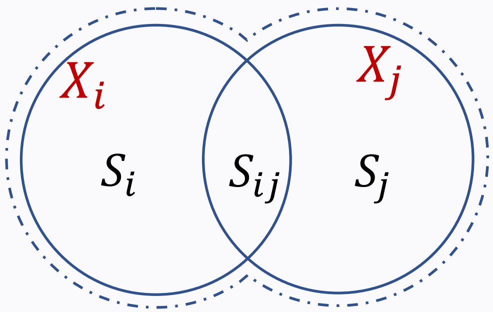
$$\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}\left[Y\middle|X_i, X_j\right]\right] = V_i + V_j + V_{ij}$$

....

$$\frac{\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[Y|X_i]\right]}{\mathbb{V}\mathrm{ar}[Y]} = \frac{V_i}{V} = S_i$$

$$\frac{\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}\left[Y\middle|X_i, X_j\right]\right]}{\mathbb{V}\mathrm{ar}[Y]} = \frac{V_i}{V} + \frac{V_j}{V} + \frac{V_{ij}}{V} = S_i + S_j + S_{ij}$$

....



ANOVA vs. The Law of Total Variance

Consider X_1,

The Law of Total Variance

$$\mathbb{V}\mathrm{ar}[\mathrm{Y}] = \underline{\mathbb{V}\mathrm{ar}_{x_\mathrm{i}}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|X_1]\right]} + \underline{\mathbb{E}_{x_i}\left[\mathbb{V}\mathrm{ar}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|X_1]\right]}$$

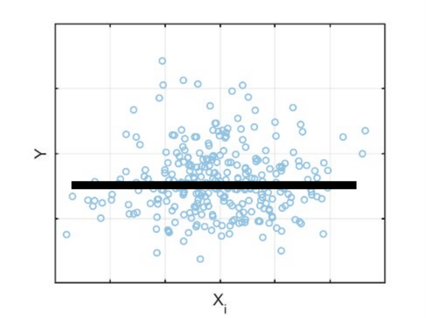
ANOVA

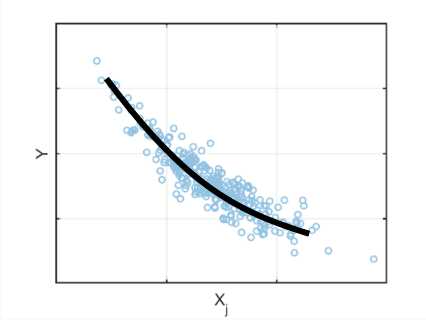
$$\mathbb{V}\mathrm{ar}[Y] = \underline{V_1} + \underline{\sum_{i=2}^{d} V_i + \sum_{i<j} V_{ij} + \ldots + V_{12..d}}$$

$$= \mathbb{V}\mathrm{ar}_{x_\mathrm{i}}\left[\mathbb{E}_{\mathbf{x}_{\bar{i}}}[\mathrm{Y}|X_1]\right]$$

Remarks

- When variables are correlated
  - The Law of Total Variance <u>does not require</u> the assumption of independence
    - Intuitive interpretation still holds



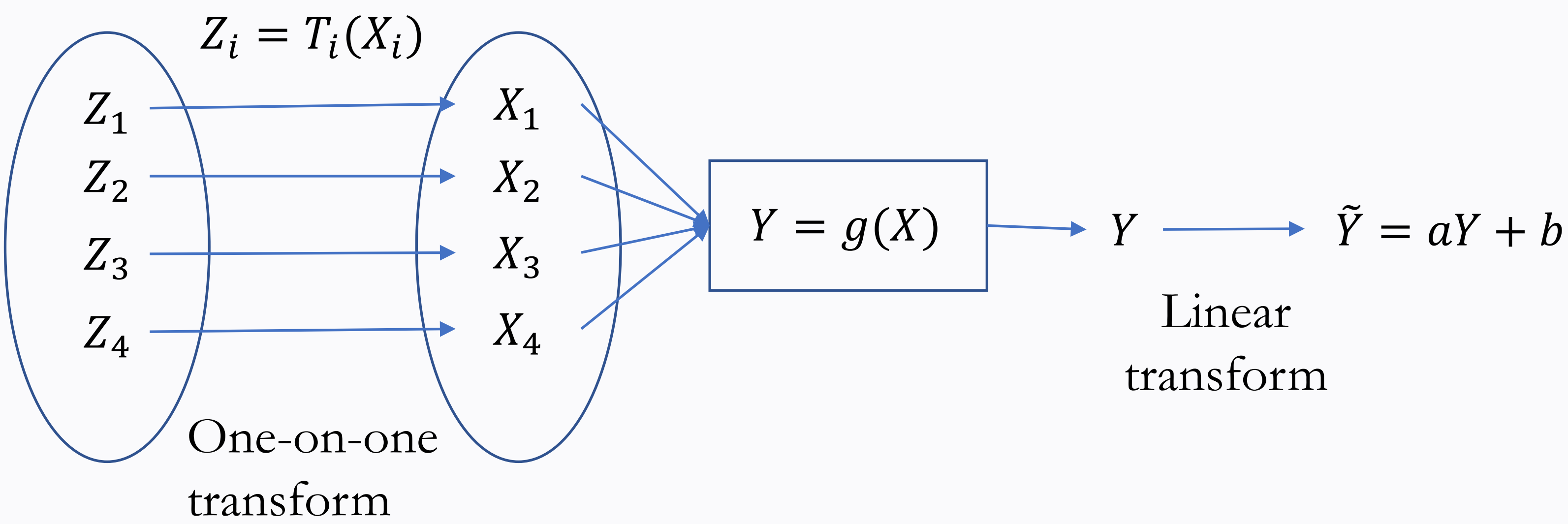


  - ANOVA <u>requires</u> the assumption of independence

$$1 > \sum_i S_i + \sum_{i<j} S_{ij} + \ldots + S_{1,2,\ldots,d}$$

Remarks

- When $\boldsymbol{X}$ are independent random variables, the sensitivity indices are invariant to any one-on-one transformation of **input** $Z_i = T_i(X_i)$



- The sensitivity indices are invariant to the linear transform of **output**

Example FORM limit state function

$$Y = G_{FORM}(\underline{z}) = \nabla G(\underline{z}^*) \cdot (\underline{z} - \underline{z}^*)$$

$$\frac{Var[E[Y|X]]}{Var[Y]}$$

$$= \frac{\partial G}{\partial z_1}(\underline{z}^*) \cdot (z_1 - z_1^*) + \frac{\partial G}{\partial u_2}(\underline{z}^*)(z_2 - z_2^*)$$

$$+ \cdots + \frac{\partial G}{\partial z_d}(\underline{z}^*)(z_d - z_d^*)$$

$z_i \sim N(0,1)$ ind.

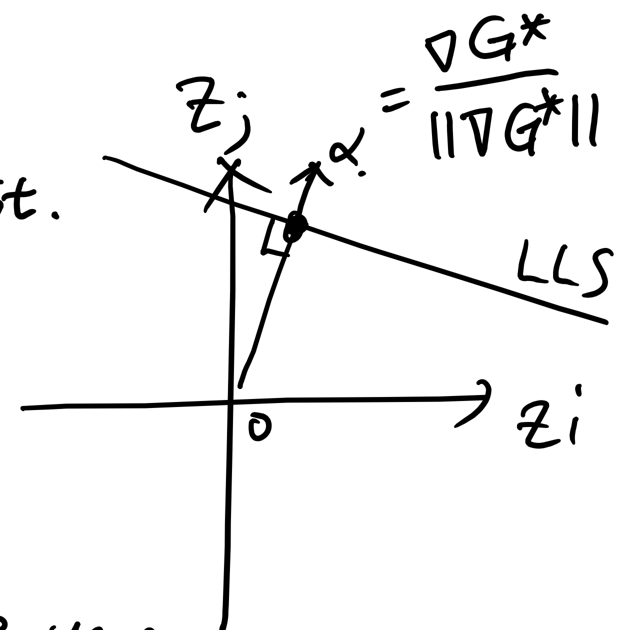
$$Var[Y] = \left(\frac{\partial G}{\partial z_1}(z^*)\right)^2 + \left(\frac{\partial G}{\partial z_2}(z^*)\right)^2 + \cdots + \frac{\partial G}{\partial z_d}(z^*)^2$$

$$= \|\nabla G^*\|^2$$

$$E[Y|z_1] = \frac{\partial G}{\partial z_1}(z^*) \cdot z_1 + \text{const.}$$

$$Var_{z_1}[E[Y|z_1]] = \left(\frac{\partial G}{\partial z_1}(z^*)\right)^2.$$

$$s_i = \frac{\left|\frac{\partial G}{\partial z_1}(z^*)\right|^2}{\|\nabla G^*\|^2} = \alpha_i^2 \leftarrow \text{Importance vector.}$$



Special case – Linear model $\boldsymbol{g(x)}$

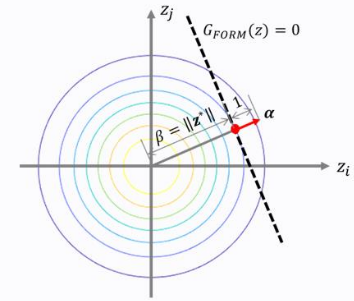
- For a linear model, below are equivalent
  - Sigma-normalized derivative
  - Linear regression coefficients
  - Variance-based sensitivity indices

Importance vector

$$\boldsymbol{\alpha} = -\frac{\nabla G(\boldsymbol{z}^*)}{\|\nabla G(\boldsymbol{z}^*)\|}$$

Note

$$\boldsymbol{\alpha} = -\frac{\boldsymbol{z}^*}{\beta}$$



- Example – FORM limit state surface

$$G_{FORM}(\boldsymbol{z}) = \nabla G(\boldsymbol{z}^*)(\boldsymbol{z} - \boldsymbol{z}^*)$$

$$= \frac{\partial G(\boldsymbol{z}^*)}{\partial z_1}(z_1 - z_1^*) + \cdots + \frac{\partial G(\boldsymbol{z}^*)}{\partial z_d}(z_d - z_d^*)$$

$$\mathbb{V}\mathrm{ar}[Y_{FORM}] = \|\nabla G(\boldsymbol{z}^*)\|^2$$

$$\mathbb{E}[Y_{FORM}|z_i] = \frac{\partial G(\boldsymbol{z}^*)}{\partial z_i} z_i$$

$$\mathbb{V}\mathrm{ar}[\mathbb{E}[Y_{FORM}|z_i]] = \left(\frac{\partial G(\boldsymbol{z}^*)}{\partial z_i}\right)^2$$

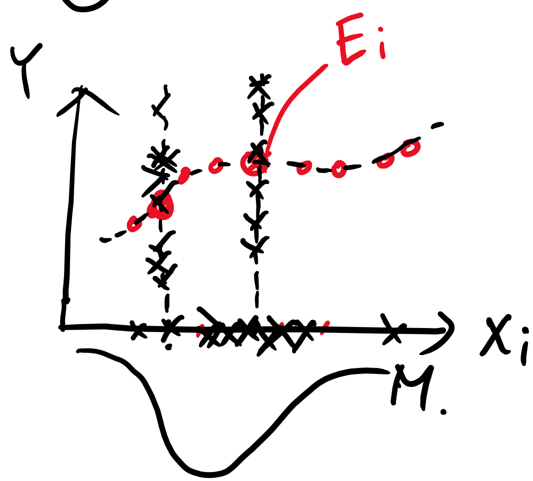
$$S_i = \frac{\mathbb{V}\mathrm{ar}[\mathbb{E}[Y_{FORM}|z_i]]}{\mathbb{V}\mathrm{ar}[Y_{FORM}]} = \alpha^2$$

Algorithms GSA.

$$S_i = \frac{Var[E[Y|X]]}{Var[Y]}$$ ← Numerator

① MCS

Samples of E_i



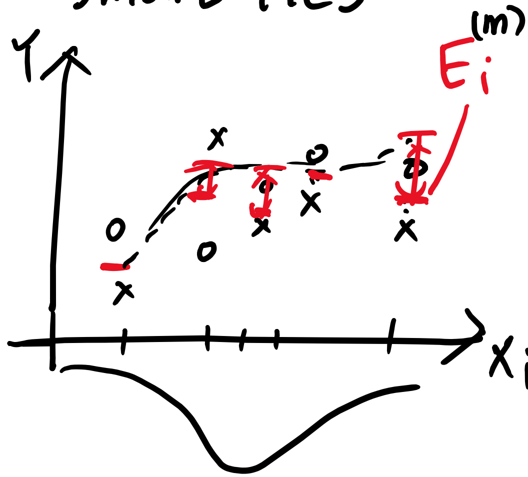
$X_{\sim i}^{(n)} \to Y^{(n)}$

$$E[Y|X] \simeq E_i = \frac{1}{N}\sum Y^{(n)}$$

$$Var[E[Y|X]] \simeq \text{sample var. of } E_i = \frac{1}{M}\sum_{m=1}^{M}(E_i - \bar{E}_i)^2$$

$\Rightarrow N \times M \times d \times 2.$

② Smart MCS

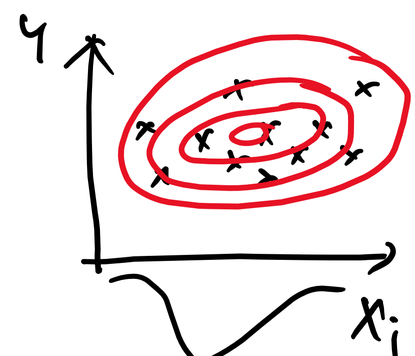


$X_{\sim i}^{A} \to Y^{A}$, "resample"

$X_{\sim i}^{A'} \to Y^{A'}$

$$Var[E[Y|X]] \simeq \text{sample var of } E_i^{(m)}$$

③ Probability model-based



fitting joint PDF of x_i, y

$$f(y|x_i) = \frac{f(x_i,y)}{f(x_i)} \Rightarrow f(x_i,y)$$

Algorithms: (1) Monte Carlo Estimation

Requires two-fold integration for "variance" and "mean" operation

$$S_i = \frac{\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\boldsymbol{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right]}{\mathbb{V}\mathrm{ar}[\mathrm{Y}]}$$

For n=1:N

sample $x_i^{(n)}$

For m=1:N

sample $\boldsymbol{x}_{\bar{i}}^{(m)}$

simulate $y^{(m,n)} = g\left(x_i^{(n)}, \boldsymbol{x}_{\bar{i}}^{(m)}\right)$

end

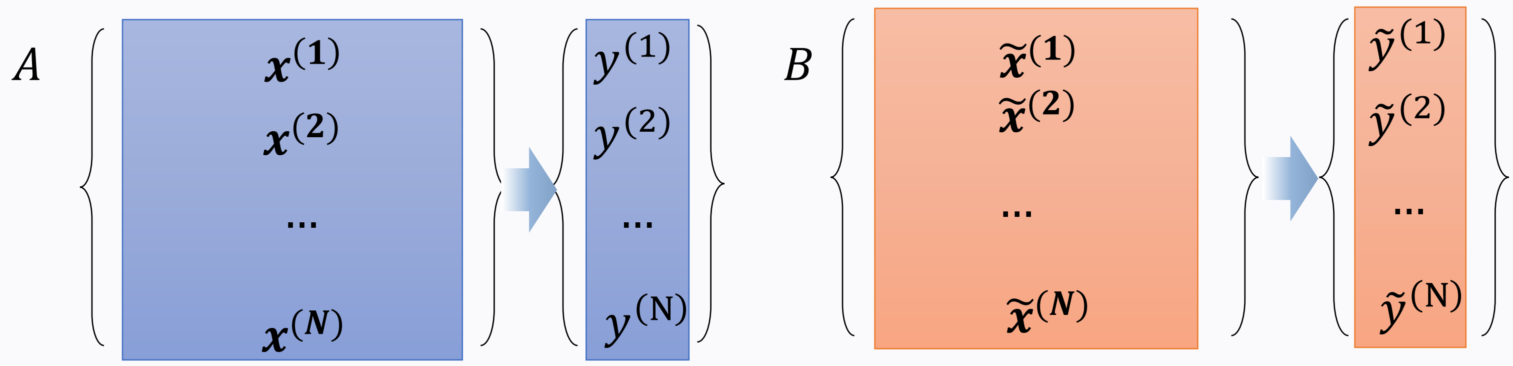
$$E^{(n)} = \mathbb{E}_{\boldsymbol{x}_{\bar{i}}}\left[\mathrm{Y}|x_i^{(n)}\right] \simeq \frac{1}{N}\sum_{m=1}^{N} y^{(m,n)}$$

end

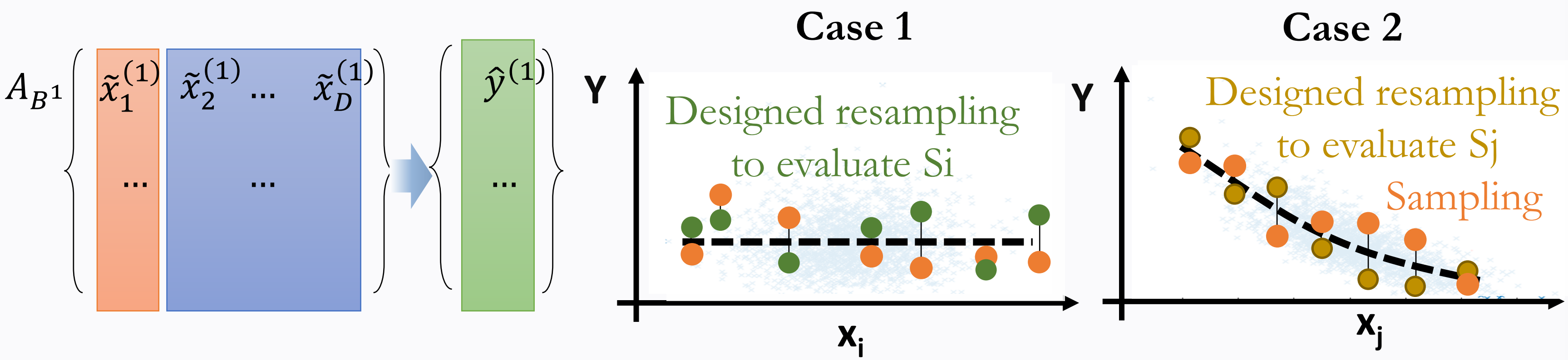
$\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{\boldsymbol{x}_{\bar{i}}}[\mathrm{Y}|x_i]\right] \simeq$ sample variance of $E^{(n)}$

Algorithms: (2) Smart Monte Carlo

- Start with two random N sample set



- Designed sample set to estimate Sobol indices of X_1



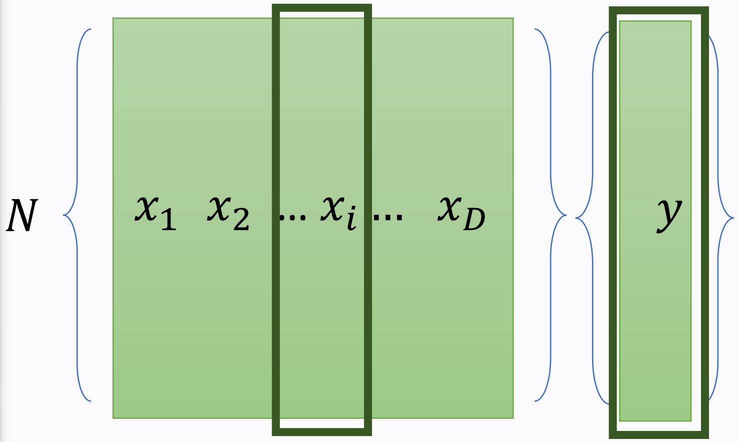
Algorithms: (2) Smart Monte Carlo

Saltelli, 2009

$V_{X_i}(E_{\mathbf{X}_{\sim i}}(Y\|X_i))$ for S_i	Reference
(a) $\frac{1}{N}\sum_{j=1}^{N} f(\mathbf{A})_j f(\mathbf{B}_{\mathbf{A}}^{(i)})_j - f_0^2$	'Sobol' 1993' [37]
(b) $\frac{1}{N}\sum_{j=1}^{N} f(\mathbf{B})_j (f(\mathbf{A}_{\mathbf{B}}^{(i)})_j - f(\mathbf{A})_j)$	[this paper]
(c) $V(Y) - \frac{1}{2N}\sum_{j=1}^{N} (f(\mathbf{B})_j - f(\mathbf{A}_{\mathbf{B}}^{(i)})_j)^2$	'Jansen 1999' [14]
$E_{\mathbf{X}_{\sim i}}(V_{X_i}(Y \mid \mathbf{X}_{\sim i}))$ for S_{Ti}	
(d) $V(Y) - \frac{1}{N}\sum_{j=1}^{N} f(\mathbf{A})_j f(\mathbf{A}_{\mathbf{B}}^{(i)})_j + f_0^2$	'Homma 1996' [11]
(e) $\frac{1}{N}\sum_{j=1}^{N} f(\mathbf{A})_j (f(\mathbf{A})_j - f(\mathbf{A}_{\mathbf{B}}^{(i)})_j)$	'Sobol' 2007' [39]
(f) $\frac{1}{2N}\sum_{j=1}^{N} (f(\mathbf{A})_j - f(\mathbf{A}_{\mathbf{B}}^{(i)})_j)^2$	'Jansen 1999' [14] and [this paper]

Algorithms: (3) Probability model-based GSA

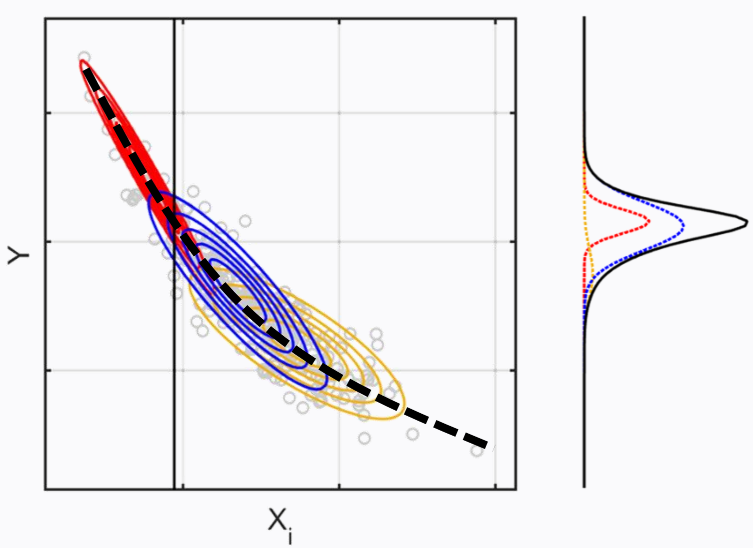
- N-MCS samples are required - existing samples can be used!



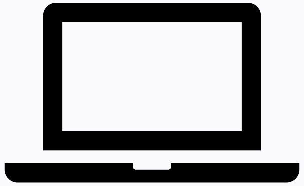
Estimation algorithm

- Approximate joint distribution of $f(X_i, Y)$ using a Gaussian mixture model (GMM)
- Estimate $\mathbb{E}[Y|X_i]$ from GMM $f(X_i, Y)$
- Repeat for different $X_i^{(n)}$ samples to get sample variance

$$\mathbb{V}\mathrm{ar}_{x_i}\left[\mathbb{E}_{x_{\bar{i}}}\left[Y|X_i^{(n)}\right]\right]$$



For Thursday class (4/21)



Bring laptop

Download quoFEM

Register for a DesignSafe Account

- quoFEM: https://simcenter.designsafe-ci.org/research-tools/quofem-application/
- DesignSafe: https://www.designsafe-ci.org/account/register/

Variance-based Reliability Sensitivity Analysis

- Reliability-oriented sensitivity analysis
- Quantity of interest:

$$q = \mathbb{1}(G(\boldsymbol{X})) = \begin{cases} 1 & G(\boldsymbol{X}) \leq 0 \\ 0 & G(\boldsymbol{X}) > 0 \end{cases}$$

Bernoulli

$P(q=1) = P_f$

$P(q=0) = 1-P_f$

Variance [edit]

The variance of a Bernoulli distributed X is

$$\mathrm{Var}[X] = pq = p(1-p)$$

- $E[q] = E[\mathbb{1}(G(\boldsymbol{X}))] = P_f$
- $Var[q] = Var[\mathbb{1}(G(\boldsymbol{X}))] = P_f(1-P_f)$

Reformulation of Sobol index

- Main Sobol index

$$S_i = \frac{\mathbb{V}\mathrm{ar}_{X_i}\left[\mathbb{E}_{\boldsymbol{X}_{\bar{i}}}[q|X_i]\right]}{\mathbb{V}\mathrm{ar}[q]} = \frac{\mathbb{V}\mathrm{ar}_{X_i}\left[\mathbb{E}_{\boldsymbol{X}_{\bar{i}}}[q|X_i]\right]}{P_f\left(1-P_f\right)}$$

- Similarly,

$$\mathbb{E}_{\boldsymbol{x}_{\bar{i}}}[q|X_i] = P_{f|X_i}$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}_{X_i}\left[\mathbb{E}_{\boldsymbol{x}_{\bar{i}}}[q|X_i]\right] &= \mathbb{V}\mathrm{ar}_{X_i}\left[P_{f|X_i}\right] \\ &= \mathbb{E}_{X_i}\left[P_{f|X_i}^2\right] - \mathbb{E}_{X_i}\left[\mathbb{E}_{\boldsymbol{X}_{\bar{i}}}\left[P_{f|X_i}\right]\right]^2 \\ &= \mathbb{E}_{X_i}\left[P_{f|X_i}^2\right] - P_f^2\end{aligned}$$

$$S_i = \frac{\mathbb{V}\mathrm{ar}_{X_i}\left[\mathbb{E}_{\boldsymbol{X}_{\bar{i}}}[q|X_i]\right]}{P_f\left(1-P_f\right)} = \frac{\mathbb{E}_{X_i}\left[{\color{red}P_{f|X_i}^2}\right] - P_f^2}{P_f\left(1-P_f\right)}$$

Reformulation of Sobol index

$$S_i = \frac{\mathbb{E}_{X_i}\left[P_{f|X_i}^2\right] - P_f^2}{P_f\left(1 - P_f\right)}$$

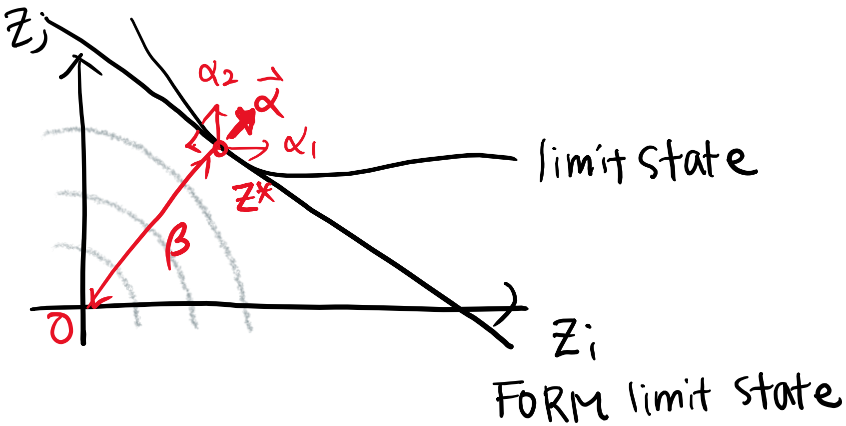
- P_f is the solution of reliability analysis
- How about $P_{f|X_i}$?

Two different combination of **Reliability Analysis and Variance Based Sensitivity analysis**:

1. Sobol indices as "by-product" of reliability analysis
   - After FORM reliability analysis
   - After sampling-based reliability analysis

2. Get Sobol indices "before" running reliability analysis
   - Probability model-based GSA

Review of FORM - β and $\boldsymbol{\alpha}$



$\vec{\alpha}$: importance vector

$$\vec{\alpha} = \frac{\nabla G(z^*)}{\|\nabla G(z^*)\|}, \quad \|\alpha\| = 1 .$$

$$\alpha_1^2 + \alpha_2 + \cdots \alpha_d = 1 .$$

β : reliability index

$$P_f = \Phi(-\beta)$$

$$\beta = \|z^*\|$$

$$\beta = \vec{\alpha} \cdot z^* = \|\alpha\| \|z^*\| = 1 \cdot \beta .$$

Given $\beta, \vec{\alpha}$

$f(z) \sim$ standard normal.

$$G(z) = \nabla G(z^*)(z - z^*) = 0 .$$

$$\vec{\alpha}(z - z^*) = 0 \qquad \vec{\alpha} z - \vec{\alpha} z^* = \vec{\alpha} z - \beta = 0$$

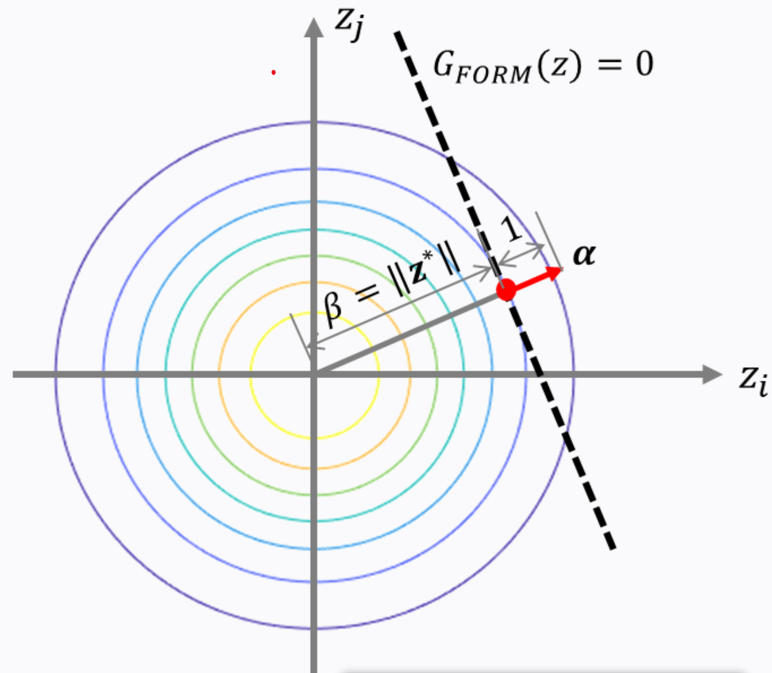
FORM and Variance-based Sensitivity Analysis

- FORM limit state

$$G_{FORM}(\boldsymbol{z}) = \nabla G(\boldsymbol{z}^*)(\boldsymbol{z} - \boldsymbol{z}^*)$$

or

$$G_{FORM}(\boldsymbol{z}) = \beta - \boldsymbol{\alpha z}$$



$$S_i = \frac{\mathbb{E}_{X_i}\left[P_{f|X_i}^2\right] - P_f^2}{P_f\left(1 - P_f\right)}$$

<u>Goal: to derive $\boldsymbol{S_i}$ in terms of $\boldsymbol{\alpha_i}$ and $\boldsymbol{\beta}$</u>

$$P_f = \mathbb{P}(\boldsymbol{\alpha Z} \geq \beta) = \mathbb{P}(\alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_d Z_d \geq \beta) = \mathbb{P}\left(\tilde{Z} \geq \beta\right) = \Phi(-\beta)$$

Standard normal

FORM and Variance-based Sensitivity Analysis

Goal: to derive $\boldsymbol{S_i}$ in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$P_f = \mathbb{P}(\boldsymbol{\alpha}\mathbf{Z} \geq \beta) = \mathbb{P}(\alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_d Z_d \geq \beta) = \mathbb{P}\left(\tilde{Z} \geq \beta\right) = \Phi(-\beta)$$

$$P_{f|z_i} = \mathbb{P}(\boldsymbol{\alpha}_{\tilde{\imath}}\mathbf{Z}_{\tilde{\imath}} \geq \beta - \alpha_i z_i) = \mathbb{P}\left(\tilde{Z} \geq \frac{\beta - \alpha_i z_i}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right) = \Phi\left(-\frac{\beta - \alpha_i z_i}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right)$$

$$S_i = \frac{\mathbb{E}_{X_i}\left[P_{f|X_i}^2\right] - P_f^2}{P_f\left(1 - P_f\right)}$$

FORM and Variance-based Sensitivity Analysis

$$S_i = \frac{\mathbb{E}_{X_i}\left[P_{f|X_i}^2\right] - P_f^2}{P_f(1 - P_f)}$$

$$\mathbb{E}_{z_i}\left[P_{f|z_i}^2\right] = \mathbb{E}_{z_i}\left[P_{f|z_i} P_{f|z_i}\right]$$

$$= \mathbb{E}_{z_i}\left[\Phi\left(\frac{\alpha_i z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right)\Phi\left(\frac{\alpha_i z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right)\right]$$

$$= \mathbb{E}_{z_i}\left[\mathbb{P}_{\tilde{\mathbf{z}}}\left[\tilde{Z}_1 \leq \frac{\alpha_i z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right]\mathbb{P}_{\tilde{\mathbf{z}}}\left[\tilde{Z}_2 \leq \frac{\alpha_i z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right]\right]$$

$$= \mathbb{E}_{z_i}\left[\mathbb{P}_{\tilde{\mathbf{z}}}\left[\left(\tilde{Z}_1 \leq \frac{\alpha_i z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right) \cap \left(\tilde{Z}_2 \leq \frac{\alpha_i z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right)\right]\right]$$

$$= \mathbb{P}_{z_i,\tilde{\mathbf{z}}}\left[\left(\tilde{Z}_1 \leq \frac{\alpha_i Z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right) \cap \left(\tilde{Z}_2 \leq \frac{\alpha_i Z_i - \beta}{\|\boldsymbol{\alpha}_{\bar{\imath}}\|}\right)\right]$$

$$= \mathbb{P}\left[\left(\tilde{Y}_1 \leq -\beta\right) \cap \left(\tilde{Y}_2 \leq -\beta\right)\right] = \Phi_2\left(-\beta, -\beta; \alpha_i^2\right)$$

$$\tilde{Y}_1 = \tilde{Z}_1\|\boldsymbol{\alpha}_{\bar{\imath}}\| - \alpha_i Z_i \qquad \tilde{Y}_1 \sim N(0,1)^2$$

$$\tilde{Y}_2 = \tilde{Z}_2\|\boldsymbol{\alpha}_{\bar{\imath}}\| - \alpha_i Z_i \qquad \tilde{Y}_2 \sim N(0,1)^2 \qquad \mathrm{corr}[\tilde{Y}_1, \tilde{Y}_2] = \alpha_i^2$$

FORM and Variance-based Sensitivity Analysis

- **Main-effect Sobol index**

$$S_i = \frac{\mathbb{E}_{Z_i}\left[P_{f|Z_i}^2\right] - P_f^2}{P_f\left(1 - P_f\right)} = \frac{\Phi_2\left(-\beta, -\beta; \alpha_i^2\right) - P_f^2}{P_f\left(1 - P_f\right)} = \frac{1}{P_f\left(1 - P_f\right)} \int_0^{\alpha_i^2} \varphi_2(-\beta, -\beta; r) dr$$

$$\Phi_2(-\beta, -\beta, \|\boldsymbol{\alpha_v}\|^2) = \Phi(-\beta)^2 + \int_0^{\|\boldsymbol{\alpha_v}\|^2} \varphi_2(-\beta, -\beta, r) \mathrm{d}r \,.$$

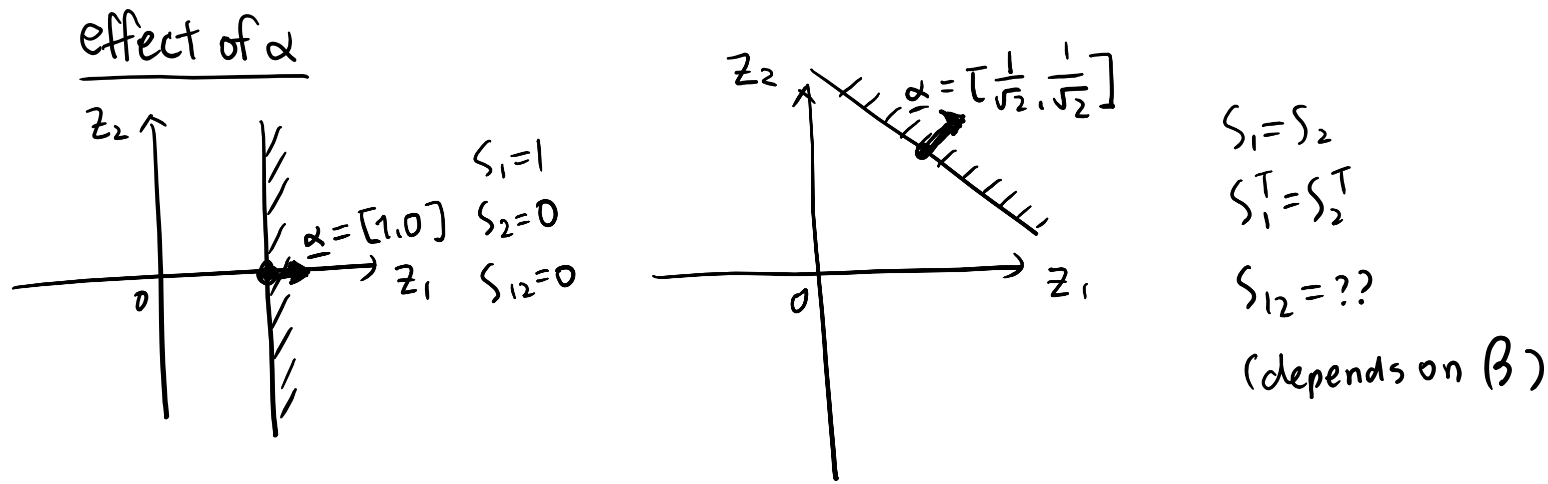
Eq. (32) in here

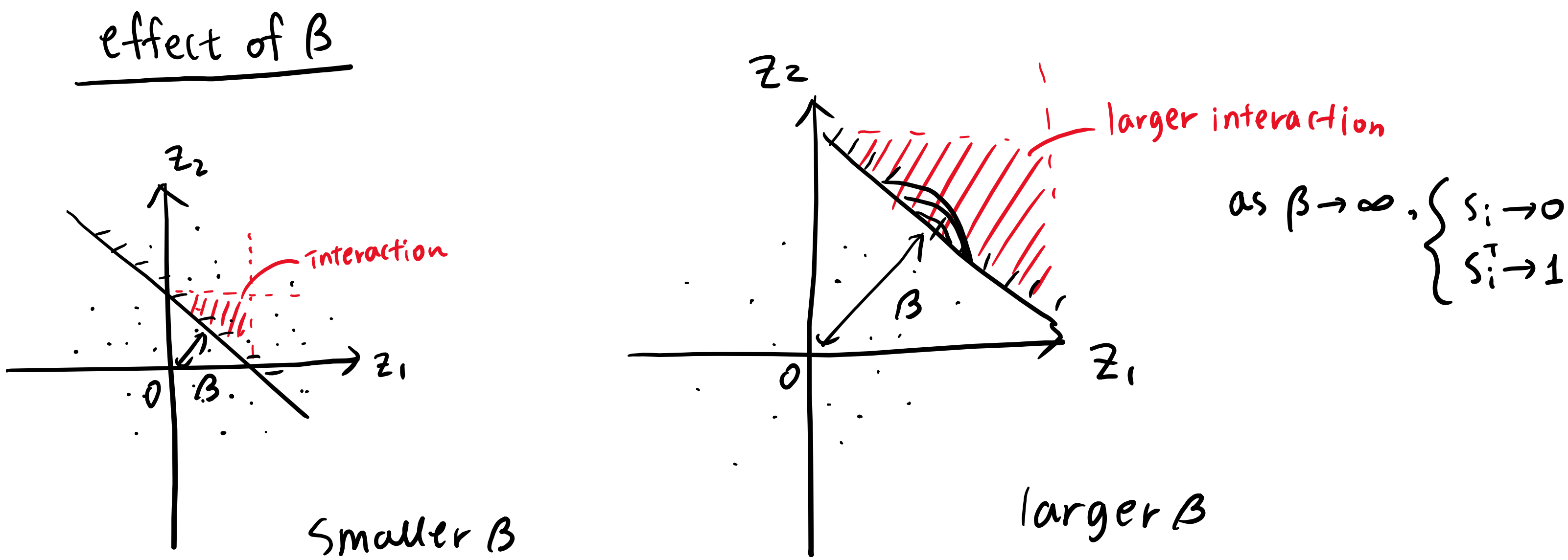
- **Total-effect Sobol index**

$$S_i^T = 1 - \frac{\mathbb{E}_{\boldsymbol{Z}_{\bar{\imath}}}\left[P_{f|\boldsymbol{Z}_{\bar{\imath}}}^2\right] - P_f^2}{P_f\left(1 - P_f\right)} = 1 - \frac{\Phi_2(-\beta, -\beta; \|\boldsymbol{\alpha}_{\bar{\imath}}\|^2) - P_f^2}{P_f\left(1 - P_f\right)}$$

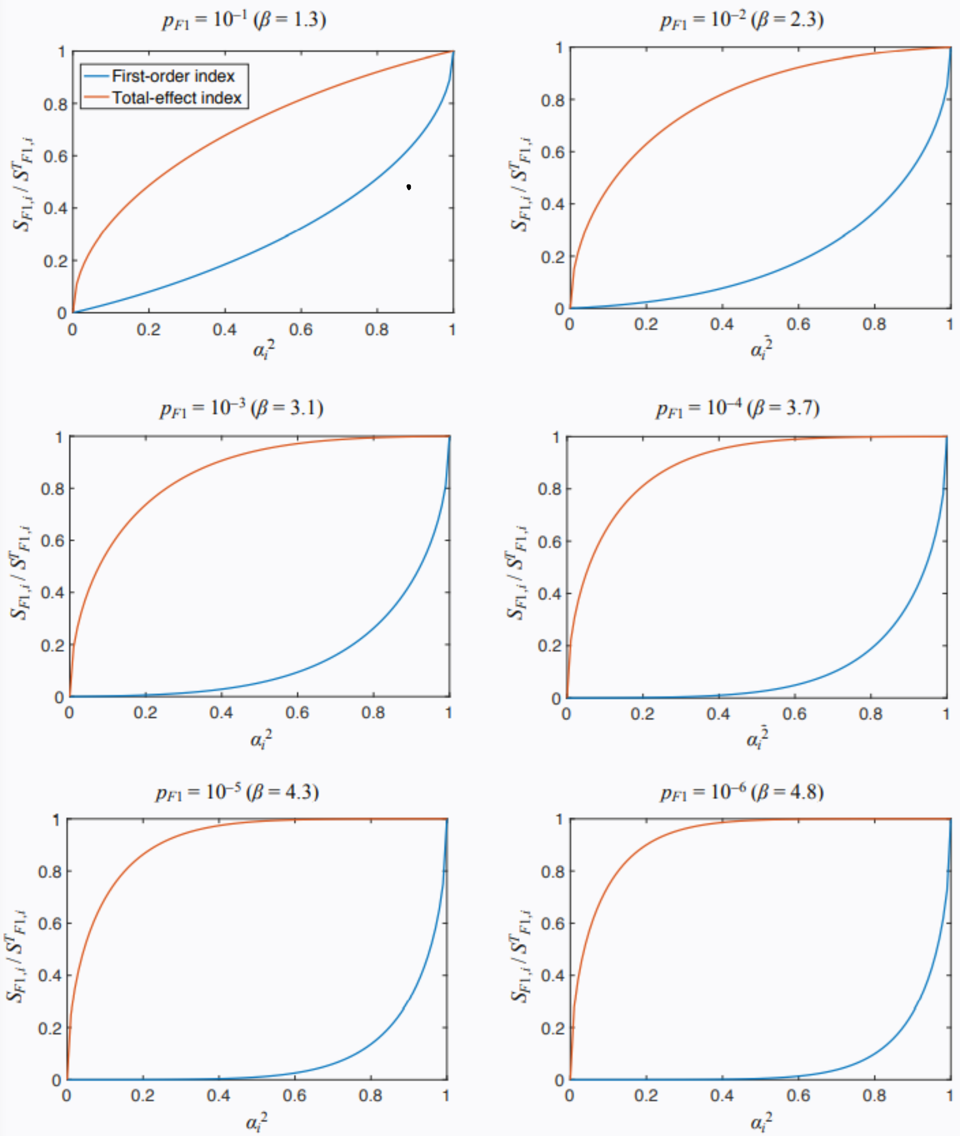
$$= \frac{1}{P_f\left(1 - P_f\right)} \int_{1-\alpha_i^2}^{1} \varphi_2(-\beta, -\beta; r) dr$$

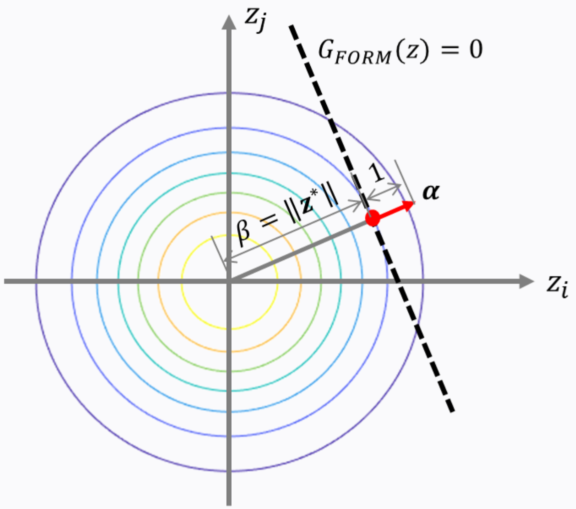
Example with two Random Variables





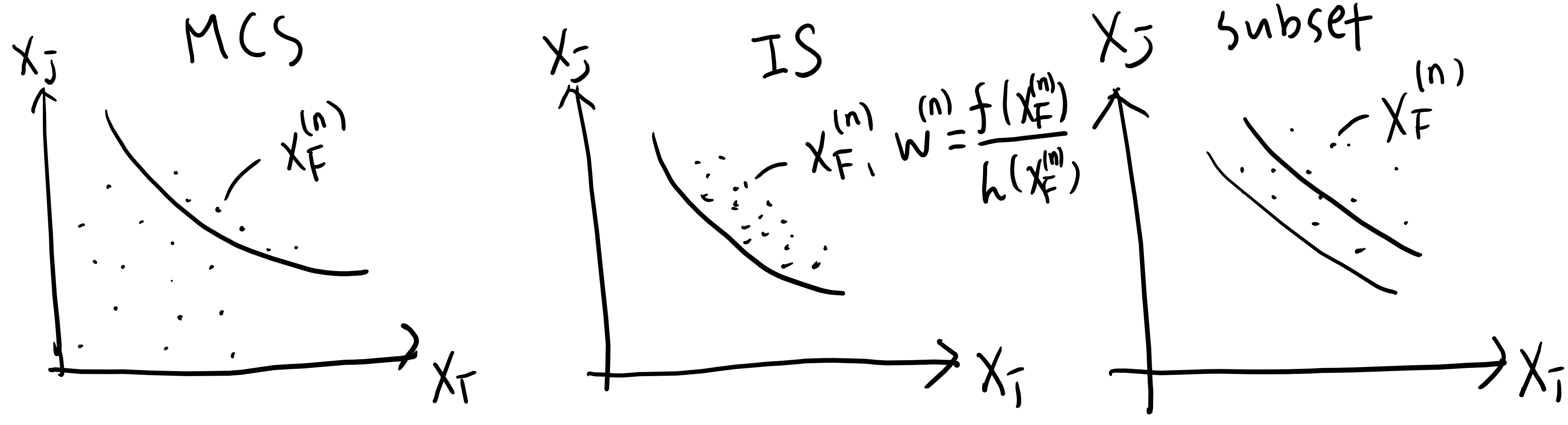
FORM and Variance-based Sensitivity Analysis





(Papaioannou and Straub, 2021)

Sampling-based Reliability Analysis and S_i



$x_F^{(n)} \sim f(\underline{x} \mid F)$

(1) collect $x_F^{(n)}$ (and $w^{(n)}$)

(2) fit distribution of $f(\underline{x} \mid F)$ using (1)

(3) calculate S_i, S_i^T using $\hat{f}(\underline{x} \mid F)$

Sampling-based Reliability Analysis and S_i

- Again, reformulation of Sobol index

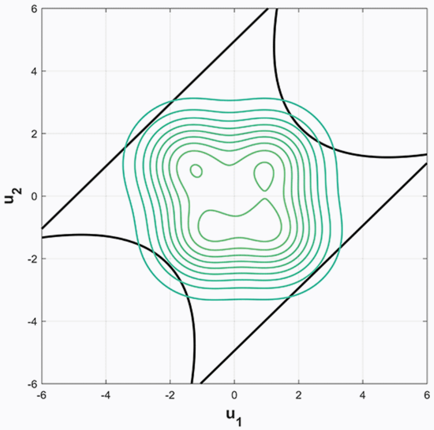
$$S_i = \frac{\mathbb{E}_{X_i}[P^2_{f|X_i}] - P_f^2}{P_f(1-P_f)}$$

$$P_{f|X_i} = \mathbb{P}(\mathcal{F}|X_i)$$

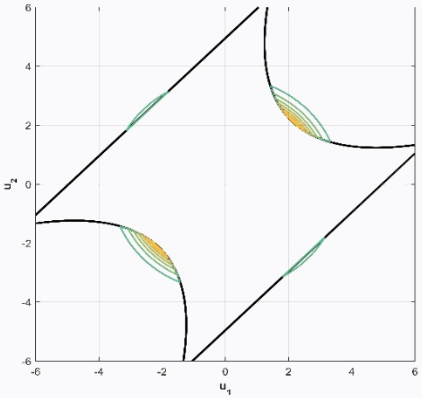
$$= \frac{\mathbb{P}(X_i|\mathcal{F})\,\mathbb{P}(\mathcal{F})}{\mathbb{P}(X_i)}$$

$$= \frac{f_{X_i|\mathcal{F}}(X_i)\, dX_i P_f}{f(X_i) dX_i}$$

$$= \frac{f_{X_i|\mathcal{F}}(X_i)\, P_f}{f(X_i)}$$



Near-optimal Density



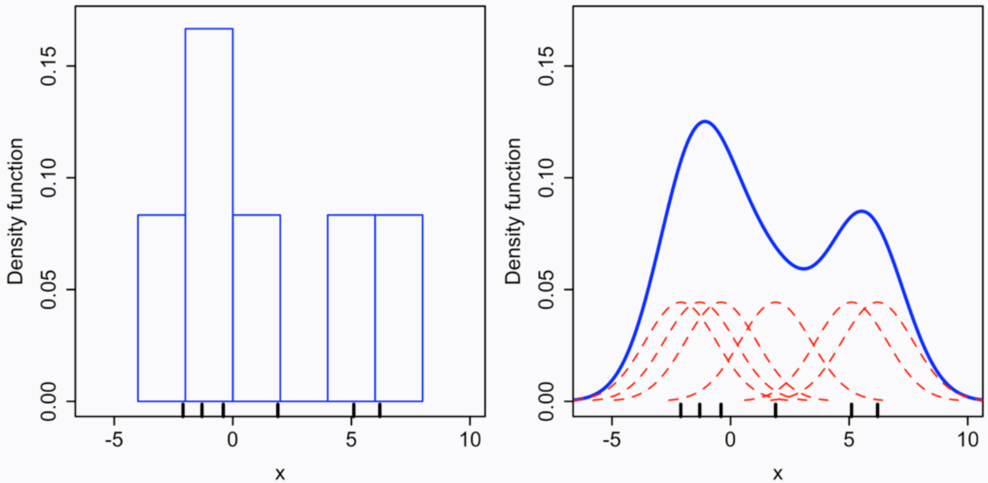
Optimal Density

Choi and Song 2017

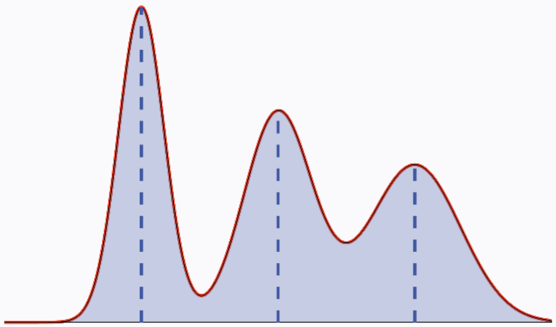
Sampling-based Reliability Analysis and S_i

- Approximation of $f_{X_i|\mathcal{F}}(X_i)$ using kernel density estimation or cross entropy-based distribution fitting

Kernel density estimation



Mixture distribution fitting

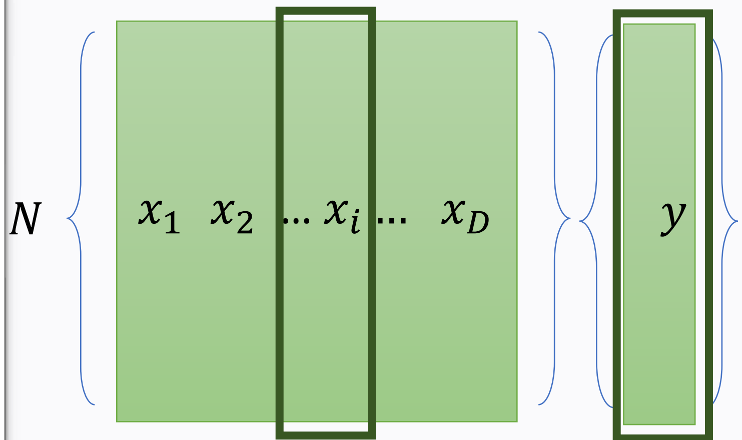


- Estimation of total-effect index is more challenging

Sensitivity Analysis before Reliability Analysis

$$S_i = \frac{\mathbb{E}_{X_i}\left[P_{f|X_i}^2\right] - P_f^2}{P_f(1 - P_f)}$$

- Probability model-based approach



- Let us define

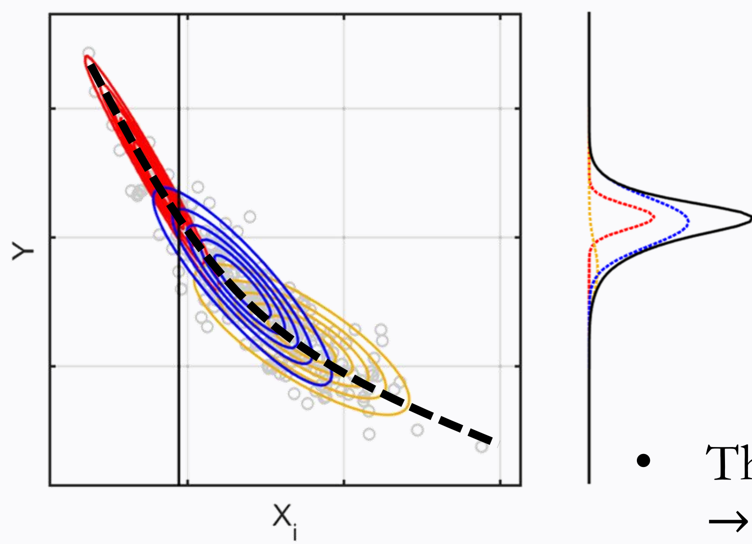
$$Y = G(\mathbf{X})$$

- Approximate joint distribution of $f(X_i, Y)$ using a Gaussian mixture model (GMM)

$$P_{f|X_i} = \mathrm{P}(\mathrm{Y} \le 0 | X_i) = \frac{f(X_i, Y \le 0)}{f(X_i)}$$

$$f(X_i, Y \le 0) = \int_{-\infty}^{0} f_{X_i,Y}(X_i, Y)\,dY$$

$$f(X_i) = \int_{-\infty}^{\infty} f_{X_i,Y}(X_i, Y)\,dY$$



- The mixture model "extrapolates" the samples → Not accurate for rare events

Toy Example

$$Y = \frac{2p}{Ebh^3}\left[(4+5\upsilon)\frac{h^2L}{4} + 2L^3\right],$$

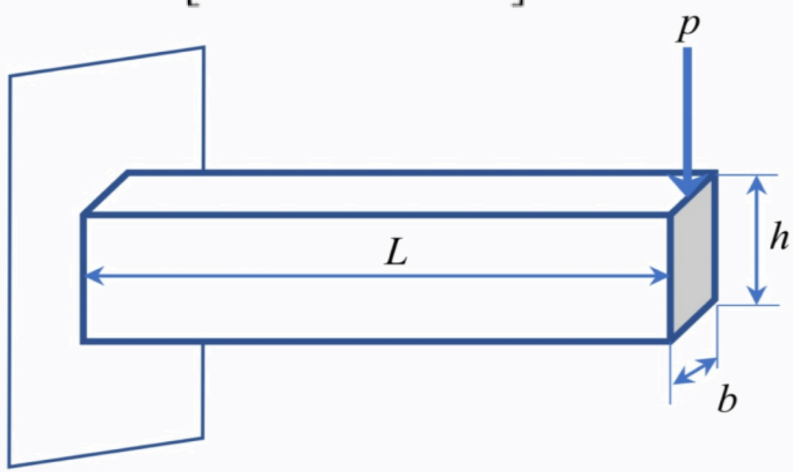
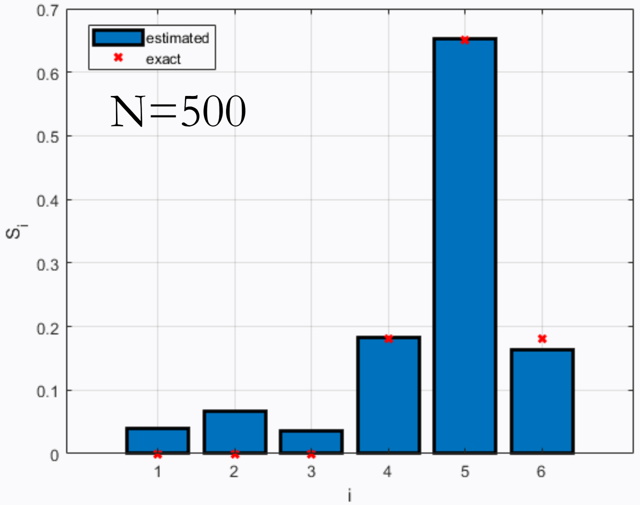


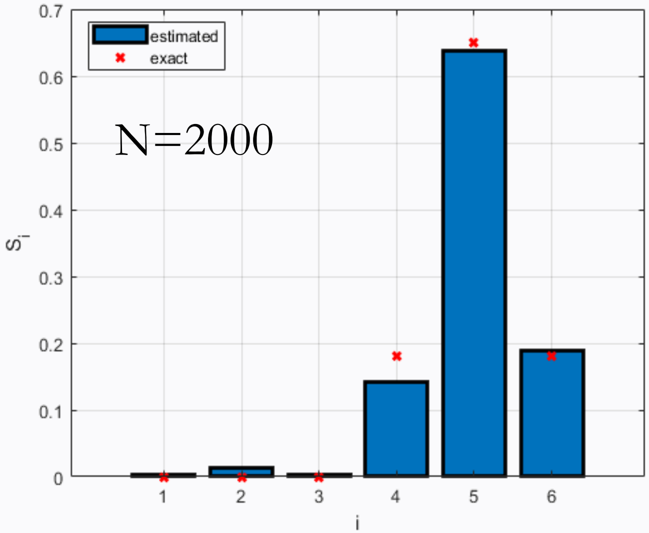
Fig. 11. A cantilever beam.

Table 1
Random variables of the cantilever beam example.

Variable	p (kN)	υ	b (m)	h (m)	L (m)
Distribution	Normal	Normal	Lognormal	Lognormal	Lognormal
Mean	65	0.225	0.2	0.3	1.5
Standard deviation	0.5	0.03	0.02	0.02	0.05

$$f_E(e) = 0.4N(e, 200, 1) + 0.6N(e, 190, 0.95^2),$$





Truss Model

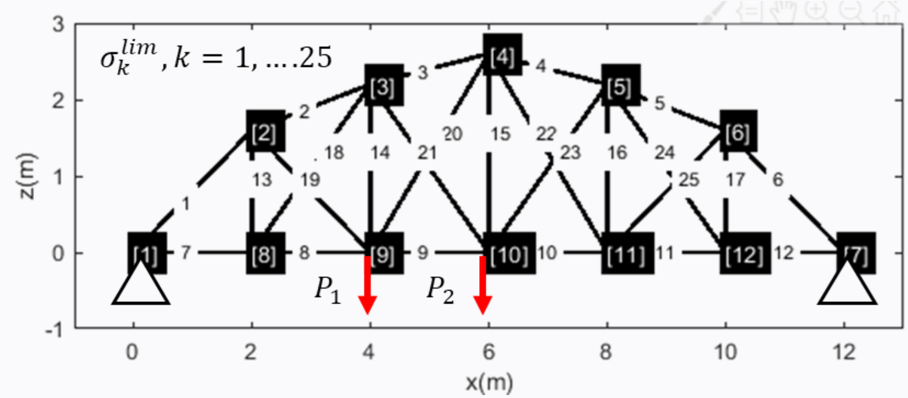
Input variables

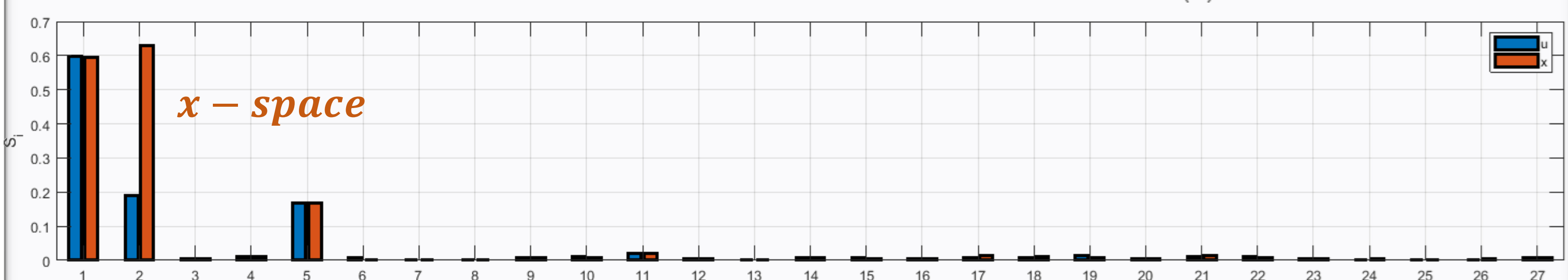
x_1, x_2: load1 (P_1) and load2 (P_1) with (correlation 0.6)

$x_3 \sim x_{27}$: strength of each member, lognormal

Output variable: limit state function

$$g(x) = \min_{k=1,\dots,25}\left(\sigma_k^{\text{thr}} - |\sigma_k(P_1, P_2)|\right)$$

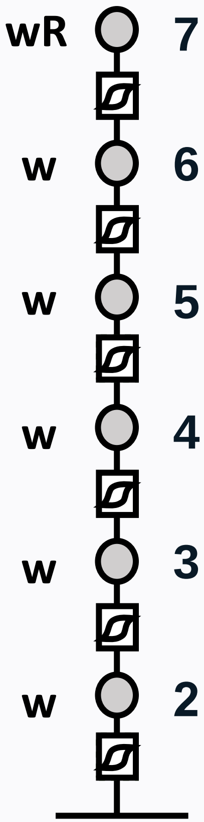


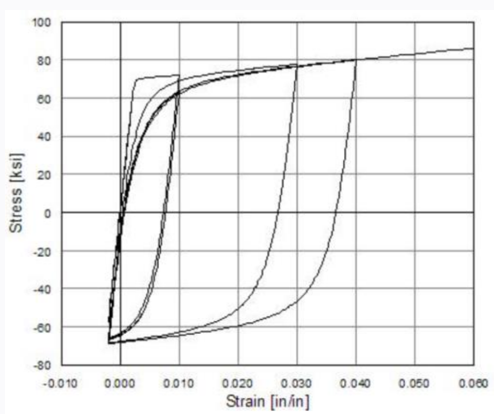


$u - space$

Examples

- Structural model: Shear building (Opensees)





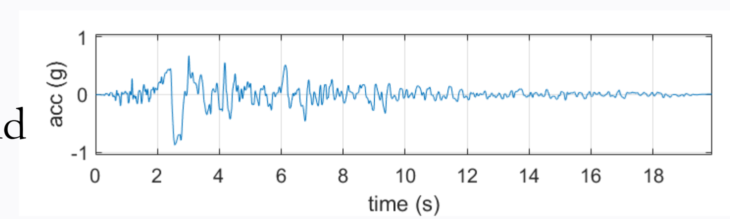
Steel 02 Material

- Input parameters

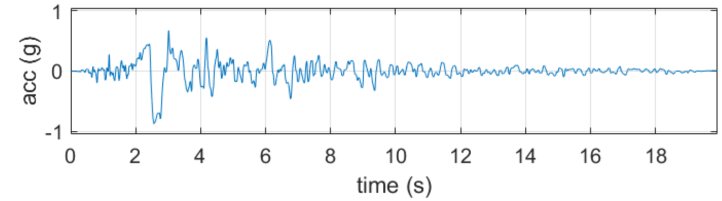
Name	Mean	C.O.V
w	100	0.1
wR	50	0.1
k	326	0.1
Fy	50	0.1
alpha	0.2	0.1
factor (PGA)	0.1	0.1

- Excitation

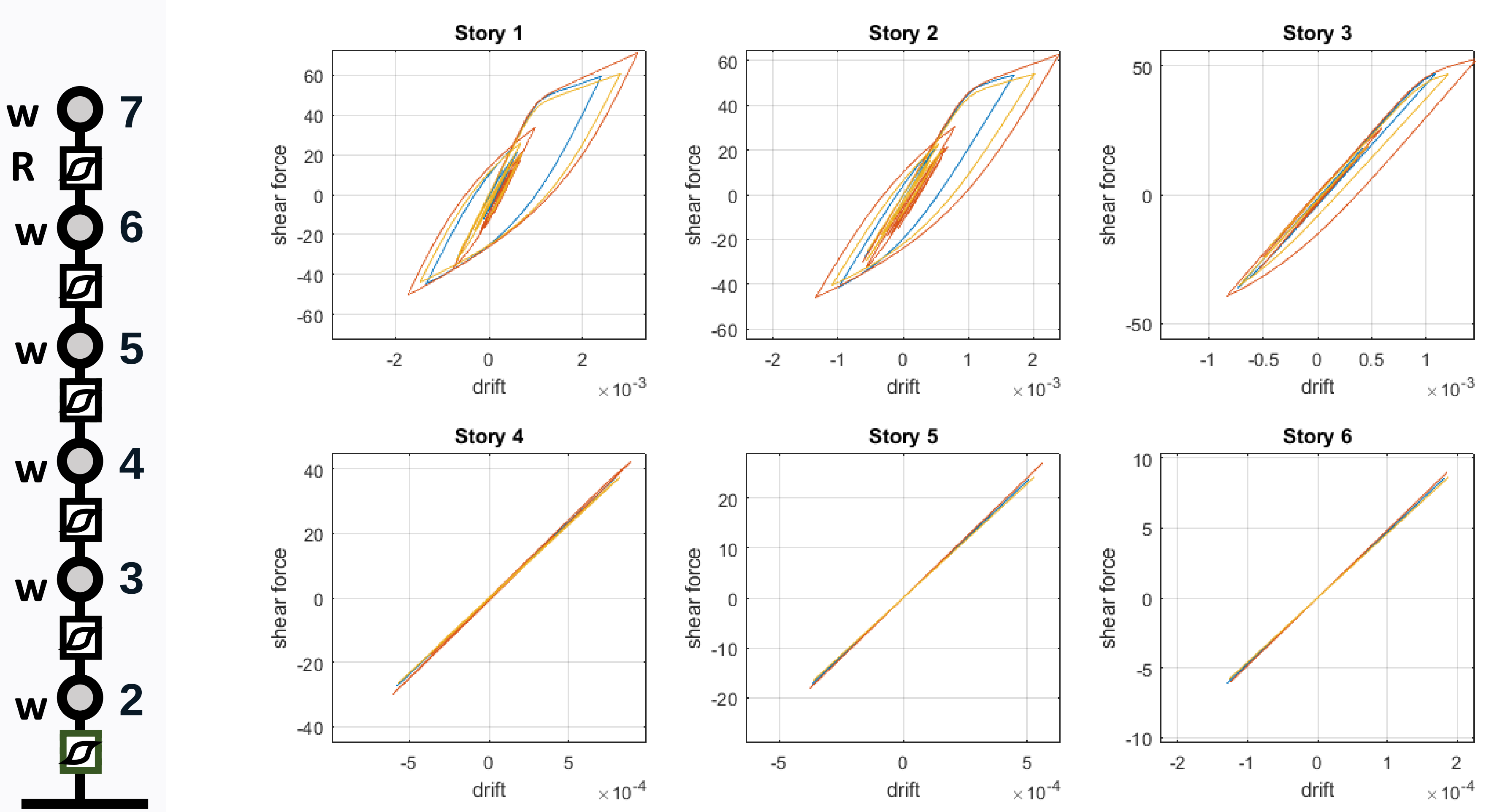
Rinaldi near-field



Nonlinear behavior



- Hysteresis curves for Rinaldi UQ



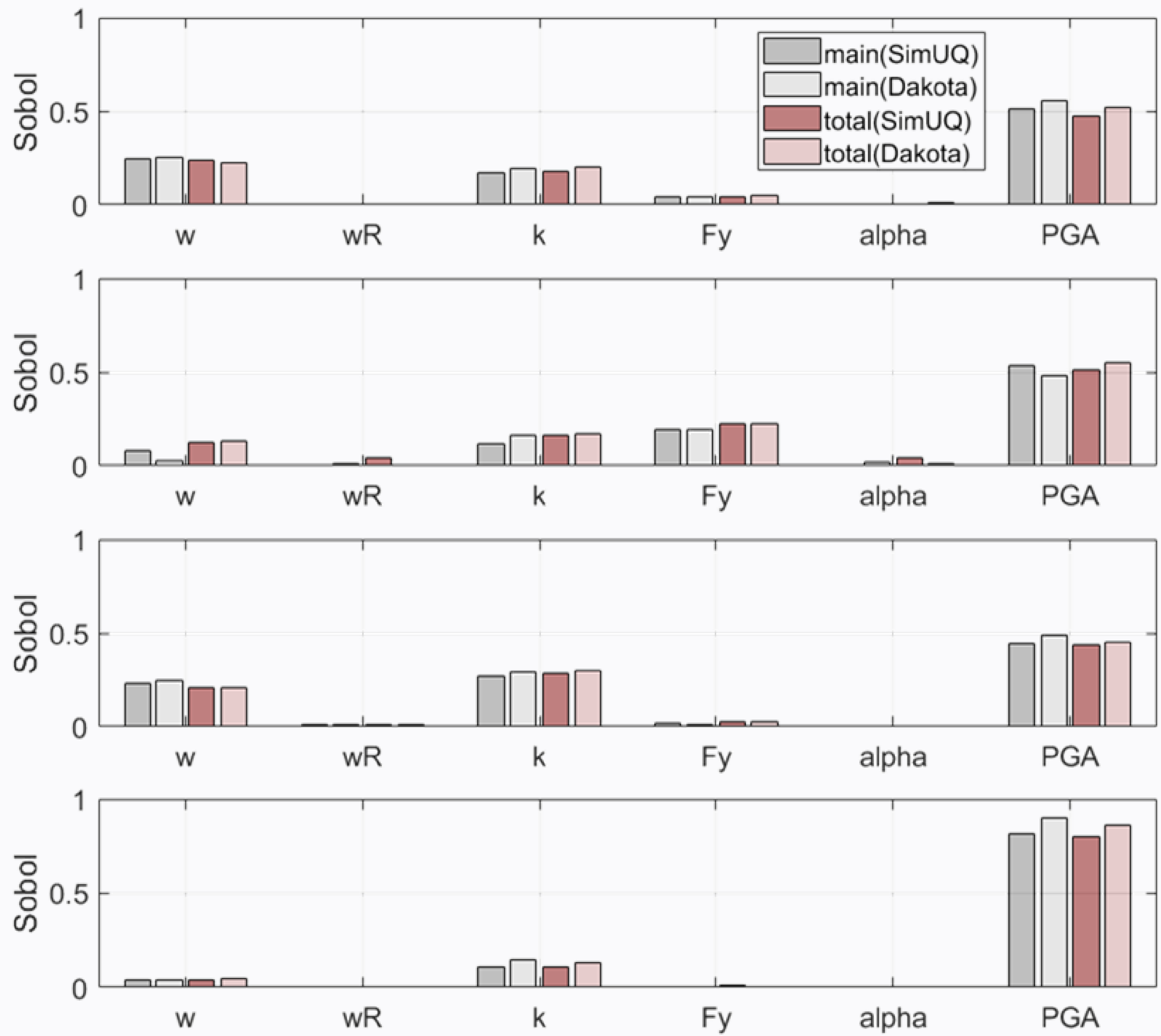
Examples

node2 disp

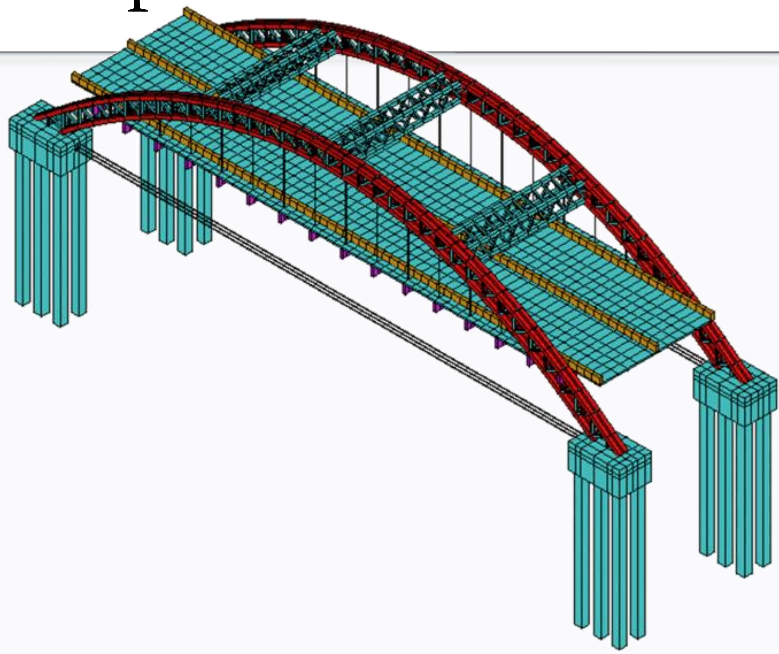
node2 acc

node7 disp

node7 acc



Examples: Parameter selection in FE model updating



GSA is performed for 15 parameters

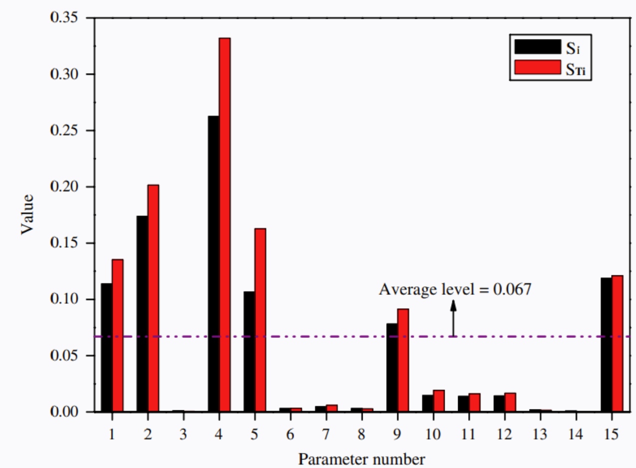


Table 4. Parameters Range of the Arch Bridge in Hertz

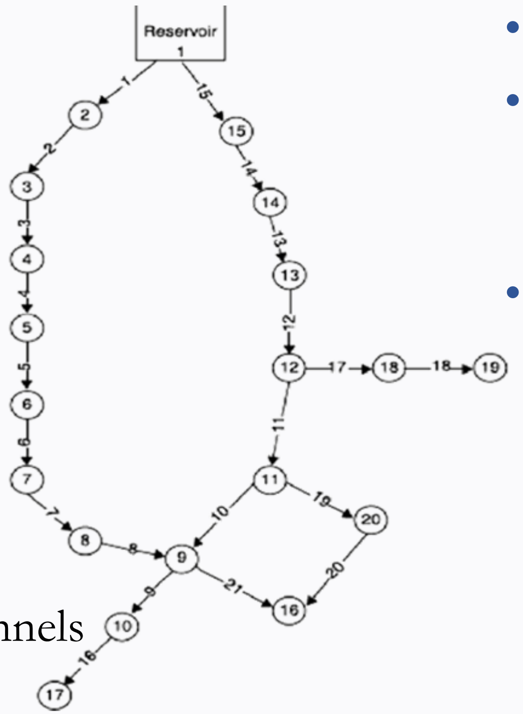
Number	Parameter	Nominal value	Lower bound	Upper bound
1	Elastic modulus of steel used in hollow tube	2.10×10^{11} Pa	1.47×10^{11} Pa	2.73×10^{11} Pa
2	Elastic modulus of concrete filled tubular arch rib	4.560×10^{10} Pa	3.192×10^{10} Pa	5.928×10^{10} Pa
3	Moment of inertia of concrete filled tubular arch rib	0.0147196 m^4	0.01030372 m^4	0.01913548 m^4
4	Density of concrete filled tubular arch rib	2,871.14 kg/m^3	2,009.798 kg/m^3	3,732.482 kg/m^3
5	Sectional area of concrete filled tubular arch rib	0.4311 m^2	0.30177 m^2	0.56043 m^2
6	Elastic modulus of wall above deck	2.850×10^{10} Pa	1.995×10^{10} Pa	3.705×10^{10} Pa
7	Density of wall above deck	2,500 kg/m^3	1,750 kg/m^3	3,250 kg/m^3
8	Elastic modulus of deck	3.0×10^{10} Pa	2.1×10^{10} Pa	3.9×10^{10} Pa
9	Density of deck	2,500 kg/m^3	1,750 kg/m^3	3,250 kg/m^3
10	Thickness of deck	0.25 m	0.2 m	0.3 m
11	Elastic modulus of cross girder	3.450×10^{10} Pa	2.415×10^{10} Pa	4.485×10^{10} Pa
12	Moment of inertia of cross girder about major axis	0.0756 m^4	0.05292 m^4	0.09828 m^4
13	Sectional area of suspender	0.002494 m^2	0.0017458 m^2	0.0032422 m^2
14	Sectional area of prestressed cable	0.0044428 m^2	0.00310996 m^2	0.00577564 m^2
15	Spring stiffness in lateral direction	5.0×10^5 N/m	3.5×10^5 N/m	6.5×10^5 N/m

"(…) Parameters 3, 6, 7, 8, 13, and 14 should be excluded from the parameter candidates because they have little influence over the objective function."

Wan, H.P. and Ren, W.X., 2015. Parameter selection in finite-element-model updating by global sensitivity analysis using Gaussian process metamodel. Journal of Structural Engineering, 141(6), p.04014164.

Examples:
Reducing the complexity of multi-objective optimization

Multi-objective Water Distribution System Optimization



New York Tunnels
Rehabilitation
(21 components)

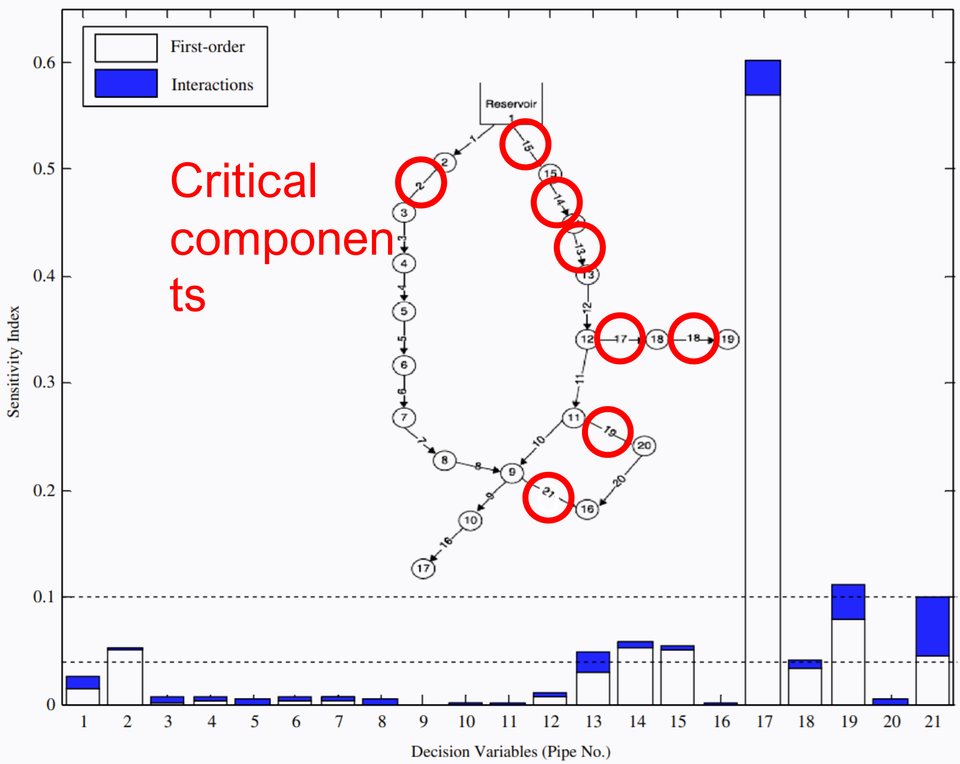
- 21 pipes in a system
- 16 Retrofitting options of each pipe :
  - 15 available diameters ranging 0.914 - 5.182 m
  - Do nothing
- Conflicting objectives:
  - Cost: capital cost (pipes, tanks, and pumps) + operating cost during a design period
  - Performance: eg. surplus power energy per unit weight

Goal: **Maximize** the performance, **minimize** the cost

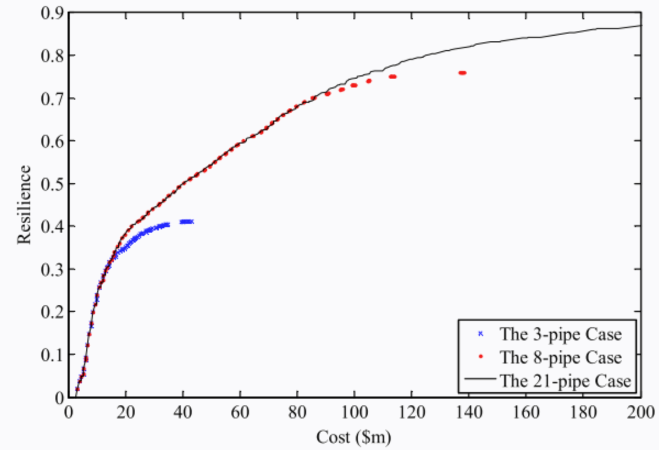
Fu, G., Kapelan, Z. and Reed, P., 2012. Reducing the complexity of multiobjective water distribution system optimization through global sensitivity analysis. Journal of Water Resources Planning and Management, 138(3), pp.196-207.

Examples: Multi-objective Water Distribution System Optimization

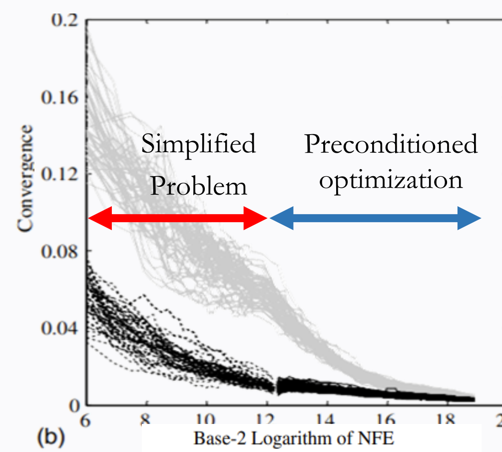
First-order and total-order indices



Best Pareto fronts



Convergence rate



Fu, G., Kapelan, Z. and Reed, P., 2012. Reducing the complexity of multiobjective water distribution system optimization through global sensitivity analysis. Journal of Water Resources Planning and Management, 138(3), pp.196-207.